CHAPTER 1

PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation is an equation involving a function of two or more variables and some of its partial derivatives. Therefore a partial differential equation contains one dependent variable and one independent variable.

Here $z$ will be taken as the dependent variable and $x$ and $y$ the independent variable so that $z = f(x, y)$.

We will use the following standard notations to denote the partial derivatives.

\[
\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t
\]

The order of partial differential equation is that of the highest order derivative occurring in it.

Formation of partial differential equation:

There are two methods to form a partial differential equation.

(i) By elimination of arbitrary constants.

(ii) By elimination of arbitrary functions.

Problems

Formation of partial differential equation by elimination of arbitrary constants:

(1) Form the partial differential equation by eliminating the arbitrary constants from $z = ax + by + a^2 + b^2$.

Solution:

Given $z = ax + by + a^2 + b^2$ ........................ (1)
Here we have two arbitrary constants $a$ & $b$.
Differentiating equation (1) partially with respect to $x$ and $y$ respectively we get

\[
\frac{\partial z}{\partial x} = a \implies p = a \\
\frac{\partial z}{\partial y} = b \implies q = a
\]

Substitute (2) and (3) in (1) we get

\[
z = px + qy + p^2 + q^2,
\]
which is the required partial differential equation.

(2) Form the partial differential equation by eliminating the arbitrary constants $a$, $b$, $c$ from

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.
\]

Solution:

We note that the number of constants is more than the number of independent variable. Hence the order of the resulting equation will be more than 1.

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we get

\[
\frac{2x}{a^2} + \frac{2y}{c^2} p = q \\
\frac{2y}{b^2} + \frac{2z}{c^2} q = 0
\]

Differentiating (2) partially with respect to $x$,

\[
\frac{1}{a^2} + \frac{1}{c^2} (zr + p^2)
\]

Where $r = \frac{\partial^2 z}{\partial x^2}$.

\[
-\frac{c^2}{a^2} = \frac{zp}{x} \\
-\frac{c^2}{a^2} = zr + p^2
\]

From (5) and (6), we get
(3) Find the differential equation of all spheres of the same radius \( c \) having their center on the yoz-plane.

Solution:

The equation of a sphere having its centre at \((0, a, b)\), that lies on the yoz-plane and having its radius equal to \( c \) is

\[
x^2 + (y - a)^2 + (z - b)^2 = c^2  \tag{1}
\]

If \( a \) and \( b \) are treated as arbitrary constants, (1) represents the family of spheres having the given property.

Differentiating (1) partially with respect to \( x \) and then with respect to \( y \), we have

\[
2x + 2(z - b) \frac{dp}{dx} = 0  \tag{2}
\]

and

\[
2(y - a) + 2(z - b) \frac{dp}{dy} = 0  \tag{3}
\]

From (2), \( z - b = \frac{xp}{\frac{dp}{dx}} \) \( \tag{4} \)

Using (4) in (3), \( y - b = \frac{qx}{\frac{dp}{dy}} \) \( \tag{5} \)

Using (4) and (5) in (1), we get

\[
x^2 + \frac{q^2 x^2}{p^2} + \frac{x^2}{p^2} = c^2
\]

i.e. \( \left(1 + p^2 + q^2\right)c^2 = c^2 p^2 \), which is the required partial differential equation.

Problems

Formation of partial differential equation by elimination of arbitrary functions:
(1) Form the partial differential equation by eliminating the arbitrary function ‘f’ from

\[ z = e^{\sigma y} f(x + by) \]

**Solution:** Given \( z = e^{\sigma y} f(x + by) \)

i.e. \( e^{-\sigma y} z = f(x + by) \) ...........................(1)

Differentiating (1) partially with respect to x and then with respect to y, we get

\[ e^{-\sigma y} p = f'(u) \] ...........................(2)

\[ e^{-\sigma y} q - ae^{-\sigma y} z = f'(u)b \] ...........................(3)

where \( u = x + by \)

Eliminating \( f'(u) \) from (2) and (3), we get

\[ \frac{q - az}{p} = b \]

i.e. \( q = az + bp \)

(2) Form the partial differential equation by eliminating the arbitrary function ‘\( \phi \)’

\[ \phi \left( z^2 - xy, \frac{x}{z} \right) = 0 \]

**Solution:** Given \( \phi \left( z^2 - xy, \frac{x}{z} \right) = 0 \) ..............................(1)

Let \( u = z^2 - xy \), \( v = \frac{x}{z} \)

Then the given equation is of the form \( \phi(u, v) = 0 \).

The elimination of \( \phi \) from equation (2), we get,

\[
\begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{vmatrix} = 0
\]
(3) Form the partial differential equation by eliminating the arbitrary function ‘f’ from

\[ z = f(2x + y) + g(3x - y) \]

**Solution:**

Given \[ z = f(2x + y) + g(3x - y) \]  
\[ \ldots \ldots \ldots (1) \]

Differentiating (1) partially with respect to x,

\[ p = f'(u)2 + g'(v)3 \]  
\[ \ldots \ldots \ldots (2) \]

Where \( u = 2x + y \) and \( v = 3x - y \)

Differentiating (1) partially with respect to y,

\[ q = f'(u)1 + g'(v)-1 \]  
\[ \ldots \ldots \ldots (3) \]

Differentiating (2) partially with respect to x and then with respect to y,

\[ r = f''(u)4 + g''(v)9 \]  
\[ \ldots \ldots \ldots (4) \]

and \[ s = f''(u)2 + g''(v)(-3) \]  
\[ \ldots \ldots \ldots (5) \]

Differentiating (3) partially with respect to y,

\[ t = f''(u)1 + g''(v)1 \]  
\[ \ldots \ldots \ldots (6) \]

Eliminating \( f''(u) \) and \( g''(v) \) from (4), (5) and (6) using determinants, we have

\[
\begin{vmatrix}
4 & 9 & r \\
2 & -3 & s \\
1 & 1 & t \\
\end{vmatrix} = 0
\]

i.e. \[ 5r + 5s - 30t = 0 \]

or \[
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = 0
\]
(4) Form the partial differential equation by eliminating the arbitrary function \( \phi \)

from

\[ z = \frac{1}{x} \phi(y - x) + \phi'(y - x). \]

**Solution:**

Given \[ z = \frac{1}{x} \phi(u) + \phi'(u) \]  
……………………(1)

Where \( u = y - x \)

Differentiating partially with respect to \( x \) and \( y \), we get

\[ p = \frac{1}{x} \phi'(u)1 - \frac{1}{x^2} \phi(u) - \phi''(u)1 \]  
………………(2)

\[ q = \frac{1}{x} \phi'(u)1 + \phi''(u)1 \]  
………………(3)

\[ r = \frac{1}{x} \phi''(u)1 + \frac{2}{x^2} \phi'(u) + \frac{2}{x^2} \phi(u) + \phi''(u)1 \]  
………………(4)

\[ s = -\frac{1}{x^2} \phi'(u) - \frac{1}{x} \phi''(u) + \phi''(u)1 \]  
………………(5)

\[ t = \frac{1}{x} \phi''(u)1 + \phi''''(u)1 \]  
………………(6)

From (4) and (6), we get

\[ r - t = \frac{2}{x^2} \phi'(u)1 - \frac{2}{x^2} \phi(u) \]

\[ = \frac{2}{x^2} \left\{ \frac{1}{x} \phi(u) + \phi'(u) \right\} \]

\[ = \frac{2}{x^2} z \]

i.e. \[ x^2 \left( \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} \right) = 2z \]

**Solutions of partial differential equations**

Consider the following two equations

\[ z = ax + by \]  
………………(1)
and \[ z = xf\left(\frac{y}{x}\right) \] \hspace{1cm} \text{........(2)}

Equation (1) contains arbitrary constants \( a \) and \( b \), but equation (2) contains only one arbitrary function \( f \).

If we eliminate the arbitrary constants \( a \) and \( b \) from (1) we get a partial differential equation of the form \( xp + yq = z \). If we eliminate the arbitrary function \( f \) from (2) we get a partial differential equation of the form \( xp + yq = z \).

Therefore for a given partial differential equation we may have more than one type of solutions.

**Types of solutions:**

(a) A solution in which the number of arbitrary constants is equal to the number of independent variables is called **Complete Integral** (or) **Complete solution**.

(b) In complete integral if we give particular values to the arbitrary constants we get **Particular Integral**.

(c) The equation which does not have any arbitrary constants is known as **Singular Integral**.

**To find the general integral:**

Suppose that \( f(x, y, z, p, q) = 0 \) \hspace{1cm} \text{............(1)}

is a first order partial differential equation whose complete solution is

\[ \phi(x, y, z, a, b) = 0 \] \hspace{1cm} \text{............(2)}

Where \( a \) and \( b \) are arbitrary constants.

Let \( b = f(a) \), where ‘\( f \)’ is an arbitrary function.

Then (2) becomes

\[ \phi(x, y, z, a, f(a)) = 0 \] \hspace{1cm} \text{............(3)}

Differentiating (3) partially with respect to ‘\( a \)’, we get

\[ \frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} \cdot f'(a) = 0 \] \hspace{1cm} \text{............(4)}
The eliminant of ‘a’ between the two equations (3) and (4), when it exists, is called the general integral of (1).

**Methods to solve the first order partial differential equation:**

**Type 1:**

**Equation of the form** \( f(p, q) = 0 \)  
\[ \text{.........(1)} \]

i.e the equation contains \( p \) and \( q \) only.

Suppose that  
\[ z = ax + by + c \]  
\[ \text{.........(2)} \]

is a solution of the equation

\[ \frac{\partial z}{\partial x} = a, \frac{\partial z}{\partial y} = b \]

\[ \Rightarrow p = a, q = b \]

substitute the above in (1), we get

\[ f(a, b) = 0 \]

on solving this we can get \( b = \phi(a) \), where \( \phi \) is a known function.

Using this value of \( b \) in (2), the complete solution of the given partial differential equation is

\[ z = ax + \phi(a)y + c \]  
\[ \text{.........(3)} \]

is a complete solution,

To find the singular solution, we have to eliminate ‘a’ and ‘c’ from

\[ z = ax + \phi(a)y + c \]

Differentiating the above with respect to ‘a’ and ‘c’, we get

\[ 0 = x + \phi'(a)y, \]

and \( 0=1 \).

The last equation is absurd. Hence there is no singular solution for the equation of Type 1.

**Problems:**

(1) Solve  
\[ p^2 + q^2 = 1. \]
Solution:

Given: \( p^2 + q^2 = 1 \)  

(1)

Equation (1) is of the form \( f(p, q) = 0 \).

Assume \( z = ax + by + c \)  

be the solution of equation (1).

From (2) we get \( p = a, q = b \).

(1) \( \Rightarrow a^2 + b^2 = 1 \)

\( \Rightarrow b = \pm \sqrt{1 - a^2} \)  

(3)

Substitute (3) in (2) we get

\( z = ax \pm \sqrt{1 - a^2} y + c \)  

(4)

This is a complete solution.

To find the general solution:

We put \( c = f(a) \) in (4), where ‘f’ is an arbitrary function.

i.e. \( z = ax \pm \sqrt{1 - a^2} y + f(a) \)  

(5)

Differentiating (5) partially with respect to ‘a’, we get

\[ x \pm \frac{a}{\sqrt{1 - a^2}} y + f'(a) = 0 \]  

(6)

Eliminating ‘a’ between equations (5) and (6), we get the required general solution.

To find the singular solution:

Differentiate (4) partially with respect to ‘a’ and ‘c’, we get

\[ 0 = x \pm \frac{a}{\sqrt{1 - a^2}}, \]

\[ 0 = 1 \text{ (which is absurd)} \]

so there is no singular solution.

(2) Solve \( p + q = pq \)
Solution:

Given:  \( p + q = pq \)  

Equation (1) is of the form  \( f(p, q) = 0 \)

Assume  \( z = ax + by + c \)

be the solution of equation (1).

From (2) we get  \( p = a, q = b \)

(1)  \( \Rightarrow a + b = ab \)

\[ \Rightarrow b = \frac{a}{a - 1} \]  

\( \ldots \ldots \ldots (3) \)

Substituting (3) in (2), we get

\[ z = ax + \frac{a}{a - 1} y + c \]  

\( \ldots \ldots \ldots (4) \)

This is a complete solution.

To find the general solution:

We put  \( c = f(a) \) in (4), we get

\[ z = ax + \frac{a}{a - 1} y + f(a) \]  

\( \ldots \ldots \ldots (5) \)

Differentiating (5) partially with respect to ‘a’, we get

\[ x - \frac{1}{(a - 1)^2} y + f'(a) \]  

\( \ldots \ldots \ldots (6) \)

Eliminating ‘a’ between equations (5) and (6), we get the required general solution

To find the singular solution:

Differentiating (4) with respect to ‘a’ and ‘c’.

\[ 0 = x - \frac{1}{(a - 1)^2} y, \]

and 0=1 (which is absurd).

So there is no singular solution.
Type 2: (Clairaut’s type)

The equation of the form

\[ z = px + qy + f(p, q) \]  \hspace{1cm} (1)

is known as Clairaut’s equation.

Assume \( z = ax + by + c \)  \hspace{1cm} (2)

be a solution of (1).

\[ \frac{\partial z}{\partial x} = a, \frac{\partial z}{\partial y} = b \]

\[ \Rightarrow p = a, q = b \]

Substitute the above in (1), we get

\[ z = ax + by + f(a, b) \]  \hspace{1cm} (3)

which is the complete solution.

Problem:

(1) Solve \( z = px + qy + \sqrt{pq} \)

Solution:

Given: \( z = px + qy + \sqrt{pq} \)  \hspace{1cm} (1)

Equation (1) is a Clairaut’s equation

Let \( z = ax + by + c \)  \hspace{1cm} (2)

be the solution of (1).

Put \( p = a, q = b \) in (1), we get

\[ z = ax + by + \sqrt{ab} \]  \hspace{1cm} (3)

which is a complete solution.

To find the general solution:

We put \( b = f(a) \) in (3), we get

\[ z = ax + f(a)y + \sqrt{af(a)} \]  \hspace{1cm} (4)

Differentiate (4) partially with respect to ‘a’, we get
Eliminating ‘a’ between equations (4) and (5), we get the required general solution

To find singular solution,
Differentiate (3) partially with respect to ‘a’, we get

\[ 0 = x + \frac{1}{2\sqrt{ab}} b \]

\[ \Rightarrow x = -\frac{\sqrt{b}}{2\sqrt{a}} \] \hspace{1cm} ............(6)

Differentiate (3) partially with respect to ‘b’, we get

\[ 0 = y + \frac{1}{2\sqrt{ab}} a \]

\[ \Rightarrow y = -\frac{\sqrt{a}}{2\sqrt{b}} \] \hspace{1cm} ............(7)

Multiplying equation (6) and (7), we get

\[ xy = \left( -\frac{\sqrt{b}}{2\sqrt{a}} \right) \left( -\frac{\sqrt{a}}{2\sqrt{b}} \right) \]

\[ xy = \frac{1}{4} \]

\[ 4xy = 1 \]

(2) Solve \[ z = px + qy + \frac{q}{p} - p \]

Solution:

Given: \[ z = px + qy + \frac{q}{p} - p \] \hspace{1cm} ............(1)

Equation (1) is a Clairaut’s equation

Let \[ z = ax + by + c \] \hspace{1cm} ............(2)

be the solution of (1).

Put \( p = a, q = b \) in (1), we get

\[ z = ax + by + \frac{b}{a} - a \] \hspace{1cm} ............(3)
which is the complete solution.

To find the general solution:

We put \( b = f'(a) \) in (3), we get

\[
z = ax + f(a)y + \frac{f(a)}{a} - a \tag{4}
\]

Differentiate (4) partially with respect to ‘a’, we get

\[
0 = x + f'(a)y + \frac{af'(a) - f(a)}{a} - 1 \tag{5}
\]

Eliminating ‘a’ between equations (4) and (5), we get the required general solution

To find the singular solution:

Differentiate (3) partially with respect to ‘a’,

\[
0 = x - \frac{b}{a} - 1
\]

\[
\Rightarrow x = \frac{b}{a^2} + 1
\]

\[
\Rightarrow a^2x = b + 1
\]

\[
\Rightarrow b = a^2x - 1 \tag{4}
\]

Differentiate (3) partially with respect to ‘b’,

\[
0 = y + \frac{1}{a}
\]

\[
\Rightarrow a = -\frac{1}{y} \tag{5}
\]

Substituting equation (4) and (5) in equation (3), we get

\[
z = \left(-1\right)\left(x + \left(a^2x - 1\right)y + \frac{a^2x - 1}{y} - \left(-1\right)\right)
\]

\[
z = \frac{-x}{y} + a^2xy - y - a^2xy + y + \frac{1}{y}
\]
Type 3:
Equations not containing x and y explicitly, i.e. equations of the form
\[ f(z, p, q) = 0 \] ............(1)
For equations of this type, it is known that a solution will be of the form
\[ z = \phi(x + ay) \] ............(2)
Where ‘a’ is the arbitrary constant and \( \phi \) is a specific function to be found out.

Putting \( x + ay = u \), (2) becomes \( z = \phi(u) \) or \( z(u) \)

\[ \therefore \quad p = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \]
and \[ q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a \cdot \frac{dz}{du} \]

If (2) is to be a solution of (1), the values of \( p \) and \( q \) obtained should satisfy (1).

i.e. \[ f\left(z, \frac{dz}{du}, a \cdot \frac{dz}{du}\right) = 0 \] ............(3)

From (3), we get
\[ \frac{dz}{du} = \psi(z, a) \] ............(4)

Now (4) is a ordinary differential equation, which can be solved by variable separable method.

The solution of (4), which will be of the form \( g(z, a) = u + b \) or \( g(z, a)x + ay + b \), is the complete solution of (1).

The general and singular solution of (1) can be found out by usual method.

Problems:

(1) Solve \( z = p^2 + q^2 \).

Solution:
Given: \[ z = p^2 + q^2 \] \hspace{1cm} \text{(1)}

Equation (1) is of the form \( f(z, p, q) = 0 \)

Assume \( z = \phi(u) \) where, \( u = x + ay \) be a solution of (1).

\[
\begin{align*}
p &= \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \Rightarrow p = \frac{dz}{du} \hspace{1cm} \text{(2)} \\
q &= \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a \frac{dz}{du} \Rightarrow q = a \frac{dz}{du} \hspace{1cm} \text{(3)}
\end{align*}
\]

Substituting equation (2) & (3) in (1), we get

\[
z = \left( \frac{dz}{du} \right)^2 + a^2 \left( \frac{dz}{du} \right)^2
\]

\[
z = \left( \frac{dz}{du} \right)^2 (1 + a^2)
\]

\[
\left( \frac{dz}{du} \right)^2 = \frac{z}{1 + a^2}
\]

\[
\frac{dz}{du} = \frac{z \frac{1}{2}}{(1 + a^2)^{\frac{1}{2}}}
\]

By variable separable method,

\[
\frac{dz}{\sqrt{z}} = \frac{du}{(1 + a^2)^{\frac{1}{2}}}
\]

By integrating, we get

\[
\int \frac{dz}{\sqrt{z}} = \frac{1}{(1 + a^2)^{\frac{1}{2}}} \int du + c
\]

\[
2\sqrt{z} = \frac{(x + ay)}{(1 + a^2)^{\frac{1}{2}}} + c
\]

\[
\sqrt{z} = \frac{(x + ay)}{2(1 + a^2)^{\frac{1}{2}}} + \frac{c}{2}
\]

\[
\sqrt{z} = \frac{(x + ay)}{2(1 + a^2)^{\frac{1}{2}}} + k
\] \hspace{1cm} \text{..........(4)}
This is the complete solution.

To find the general solution:

We put  \( k = f(a) \) in (4), we get

\[
\sqrt{z} = \frac{(x + ay)}{2(1 + a^2)^{\frac{3}{2}}} + f(a) \quad \ldots \ldots (5)
\]

Differentiate (5) partially with respect to ‘a’, we get

\[
0 = \frac{y - xa}{2(1 + a^2)^{\frac{3}{2}}} + f'(a) \quad \ldots \ldots (6)
\]

Eliminating ‘a’ between equations (4) and (5), we get the required general solution.

To find the singular solution:

Differentiate (4) partially with respect to ‘a’ and ‘k’, we get

\[
0 = \frac{y - xa}{2(1 + a^2)^{\frac{3}{2}}} \quad \ldots \ldots (7)
\]

and  \( 0 = 1 \) (which is absurd)

So there is no singular solution.

(2) Solve \( 9(p^2 z + q^2) = 4 \).

Solution:

Given: \( 9(p^2 z + q^2) = 4 \) \ldots \ldots (1)

Equation (1) is of the form \( f(z, p, q) = 0 \)

Assume  \( z = \phi(u) \) where \( u = x + ay \) be a solution of (1).

\[
p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \Rightarrow p = \frac{dz}{du} \quad \ldots \ldots (2)
\]

\[
q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a \frac{dz}{du} \Rightarrow q = a \frac{dz}{du} \quad \ldots \ldots (3)
\]
Substituting equation (2) & (3) in (1), we get

\[ 9 \left( \frac{dz}{du} \right)^2 \left( z + a^2 \right) = 4 \]

\[ 9 \left( \frac{dz}{du} \right)^2 \left( z + a^2 \right) = 4 \]

\[ \left( \frac{dz}{du} \right)^2 = \frac{4}{9(z + a^2)} \]

\[ \frac{dz}{du} = \pm \frac{2}{3} \cdot \frac{1}{\sqrt{z + a^2}} \]

\[ \sqrt{z + a^2} \, dz = \pm \frac{2}{3} \, du \]

Integrating the above, we get

\[ \int (z + a^2)^{\frac{1}{3}} \, dz = \pm \frac{2}{3} \int du + c \]

\[ \frac{2}{3} (z + a^2)^{\frac{2}{3}} = \pm \frac{2}{3} u + c \]

\[ \frac{2}{3} (z + a^2)^{\frac{2}{3}} = \pm \frac{2}{3} (x + ay) + c \]

\[ \Rightarrow (z + a^2)^{\frac{3}{2}} = \pm (x + ay) + k \]

\[ ..........(4) \]

This is the complete solution.

To find the general solution:

We put \( k = f(a) \) in (4), we get

\[ (z + a^2)^{\frac{3}{2}} = \pm (x + ay) + f(a) \]

\[ ..........(5) \]

Differentiate (5) partially with respect to ‘a’, we get

\[ -3a(z - a^2)^{\frac{1}{2}} = \pm y + f'(a) \]

\[ ..........(6) \]

Eliminating ‘a’ between equations (4) and (5), we get the required general solution.

To find the singular solution:

Differentiate (4) partially with respect to ‘a’ and ‘k’, we get
\[ 3a\left( z + a^2 \right)^{1/2} = \pm y \]  
\[ \text{...........(7)} \]

and  
\[ 0 = 1 \]  
(which is absurd)

So there is no singular solution.

**Type 4:**

**Equations of the form**

\[ f(x, p) = g(y, q) \]  
\[ \text{...........(1)} \]

i.e. equation which do not contain \( z \) explicitly and in which terms containing \( p \) and \( x \) can be separated from those containing \( q \) and \( y \).

To find the complete solution of (1),

We assume that \[ f(x, p) = g(y, q) = a \] where ‘a’ is an arbitrary constant.

Solving \[ f(x, p) = a \], we can get \( p = \phi(x, a) \) and solving \( g(y, q) = a \), we can get \( q = \psi(y, a) \).

Now

\[ dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \text{ or } pdx + qdy \]

i.e. \[ dz = \phi(x, a)dx + \psi(y, a)dy \]

Integrating with respect to the concerned variables, we get

\[ z = \int \phi(x, a)dx + \int \psi(y, a)dy + b \]  
\[ \text{...........(2)} \]

The complete solution of (1) is given by (2), which contains two arbitrary constants ‘a’ and ‘b’.

The general and singular solution of (1) can be found out by usual method.

**Problems:**

(1) Solve \( pq = xy \).

**Solution:**

Given: \( pq = xy \)

\[ \Rightarrow \frac{p}{x} = \frac{y}{q} \]  
\[ \text{...........(1)} \]

Equation (1) is of the form \( f(x, p) = g(y, q) \).
Let \( \frac{P}{x} = \frac{y}{q} = a \) (say)

\[
\frac{P}{x} = a \quad \Rightarrow \quad p = ax \quad \text{.........(2)}
\]

Similarly, \( \frac{y}{q} = a \quad \Rightarrow \quad q = \frac{y}{a} \quad \text{.........(3)}

Assume \( dz = pdx + qdy \) be a solution of (1)

Substitute equation (2) and (3) to the above, we get

\[
dz = axdx + \frac{y}{a}dy
\]

Integrating the above we get,

\[
\int dz = a \int xdx + \frac{1}{a} \int ydy + c
\]

\[
z = \frac{ax^2}{2} + \frac{y^2}{2a} + c
\]

\[
2z = ax^2 + \frac{y^2}{a} + k \quad \text{.........(4)}
\]

This is the complete solution.

The general and singular solution of (1) can be found out by usual method.

(2) Solve \( p + q = x + y \).

Solution:

Given: \( p + q = x + y \)

\[
\Rightarrow p - x = q - y \quad \text{.........(1)}
\]

Equation (1) is of the form \( f(x, p) = g(y, q) \)

Let \( p - x = y - q = a \) (say)

\[
p - x = a \quad \Rightarrow \quad p = x + a \quad \text{.........(2)}
\]

Similarly, \( y - q = a \quad \Rightarrow \quad q = y - a \quad \text{.........(3)}
\]

Assume \( dz = pdx + qdy \) be a solution of (1)

Substitute equation (2) and (3) to the above, we get

\[
dz = (x + a)dx + (y - a)dy
\]

Integrating the above we get,
This is the complete solution.

The general and singular solution of (1) can be found out by usual method.

Equations reducible to standard types-transformations:

Type A:

Equations of the form \( f(x^n p, y^n q) = 0 \) or \( f(x^n p, y^n q, z) = 0 \).

Where \( m \) and \( n \) are constants, each not equal to 1.

We make the transformations \( x^{1-m} = X \) and \( y^{1-n} = Y \).

Then

\[
p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = (1-m)x^{-m}P, \quad \text{where } P = \frac{\partial z}{\partial X}
\]

\[
q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = (1-n)y^{-n}Q, \quad \text{where } Q = \frac{\partial z}{\partial Y}
\]

Therefore the equation \( f(x^n p, y^n q) = 0 \) reduces to \( f\{p, Q\} = 0 \), which is a type 1 equation.

The equation \( f(x^n p, y^n q, z) = 0 \) reduces to \( f\{p, Q, z\} = 0 \), which is a type 3 equation.

Problem:

(1) Solve \( p^2 x^4 + y^2 q = 2z^2 \).

Solution:

Given: \( p^2 x^4 + y^2 q = 2z^2 \)

This can be written as

\[
\left(p x^2 \right)^2 + \left(q y^2 \right)^2 = 2z^2.
\]

Which is of the form \( f(x^n p, y^n q, z) = 0 \), where \( m=2, n=2 \).
Put \( X = x^{1-m} = \frac{1}{x} \); \( Y = y^{1-n} = \frac{1}{y} \)

\[
P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = p(x^2) = -px^2
\]

\[
Q = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial Y} = q(y^2) = -qy^2
\]

Substituting in the given equation,

\[P^2 - qz = 2z^2.
\]

This is of the form \( f(p, q, z) = 0 \).

Let \( Z = f(X + aY) \), where \( u = X + aY \)

\[
P = \frac{dz}{du}, \quad Q = a \frac{dz}{du}
\]

Equation becomes,

\[
\left( \frac{dz}{du} \right)^2 - az \frac{dz}{du} - 2z^2 = 0
\]

Solving for \( \frac{dz}{du} \),

\[
\frac{dz}{du} = \frac{az \pm \sqrt{a^2 z^2 + 8z^2}}{2}
\]

\[
\frac{dz}{z} = \frac{a \pm \sqrt{a^2 + 8}}{2} \frac{1}{du}
\]

\[
\log z = \frac{a \pm \sqrt{a^2 + 8}}{2} (X + aY) + b
\]

\[
\log z = \frac{a \pm \sqrt{a^2 + 8}}{2} \left( \frac{1 + a}{x} + \frac{a}{y} \right) + b \text{ is a complete solution.}
\]

The general and singular solution can be found out by usual method.

**Type B:**

**Equations of the form** \( f(x^k, p, z^k q) = 0 \) \( \text{or} \quad f(x^k, p, z^k q, x, y) = 0 \).

Where \( k \) is a constant, which is not equal to -1.

We make the transformations \( Z = z^{k+1} \).

Then \( P = \frac{\partial Z}{\partial x} = (k + 1) x^k p \) and
Therefore the equation $f(z^k p, z^k q) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}\right) = 0$, which is a type 1 equation.

The equation $f(z^k p, z^k q, x, y) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}, x, y\right) = 0$, which is a type 4 equation.

**Problems:**

(1) **Solve:** $z^4 q^2 - z^2 p = 1$.

**Solution:**

Given: $z^4 q^2 - z^2 p = 1$.

The equation can be rewritten as $\left( z^2 q \right)^2 - \left( z^2 p \right)^2 = \ldots \ldots \ldots \ldots \ldots (1)$

Which contains $z^2 p$ and $z^2 q$.

Hence we make the transformation $Z = z^2 q$.

\[
P = \frac{\partial Z}{\partial x} = 3z^2 p
\]

\[
\Rightarrow z^2 p = \frac{P}{3}
\]

Similarly $z^2 q = \frac{Q}{3}$

Using these values in (1), we get

\[Q^2 - 3P = 9 \quad \ldots \ldots \ldots \ldots \ldots (2)\]

As (2) is an equation containing $P$ and $Q$ only, a solution of (2) will be of the form

\[Z = ax + by + c \quad \ldots \ldots \ldots \ldots \ldots (3)\]

Now $P = a$ and $Q = b$, obtained from (3) satisfy equation (2)

\[b^2 - 3a = 9\]

i.e. $b = \pm \sqrt{3a + 9}$

Therefore the complete solution of (2) is $Z = ax \pm \sqrt{3a + 9} + c$

i.e. complete solution of (1) is $z^3 = ax \pm \sqrt{3a + 9} + c$
Singular solution does not exist. General solution is found out as usual.

**Type C:**

Equations of the form \( f(x^m z^k p, y^n z^k q) = 0 \), where \( m, n \neq 1; k \neq -1 \)

We make the transformations

\[ X = x^{1-m}, Y = y^{1-n} \text{ and } Z = z^{k+1} \]

Then

\[
P = \frac{\partial Z}{\partial X} = \frac{dZ}{dz} \frac{\partial z}{\partial x} \frac{dx}{dX} = (k+1)k^k p \cdot \frac{x^m}{1-m}
\]

and

\[
Q = (k+1)k^k q \cdot \frac{y^n}{1-n}
\]

Therefore the given equation reduces to

\[
f\left(\frac{1-m}{k+1}, \frac{1-n}{k+1}\right) = 0
\]

This is of type 1 equation.

**Problem:**

(1) Solve \( z^2 \left( \frac{p^2}{x^2} + \frac{q^2}{y^2} \right) = 1 \)

**Solution:**

Given: \( z^2 \left( \frac{p^2}{x^2} + \frac{q^2}{y^2} \right) = 1 \)

It can be rewritten as \( (x^{-1} zp)^2 + (y^{-1} zq)^2 = 1 \) \…………(1)

which is of the form \( (x^m z^k p)^2 + (y^n z^k q)^2 = 1 \)

we make the transformations

\[ X = x^{1-m}, Y = y^{1-n} \text{ and } Z = z^{k+1} \]

i.e. \( X = x^2, Y = y^2 \text{ and } Z = z^2 \)

Then

\[
p = \frac{\partial z}{\partial x} = \frac{dz}{dx} \frac{\partial Z}{\partial X} \frac{dX}{dx} = \frac{1}{2z} P \cdot 2x
\]

\[ \Rightarrow P = x^{-1} zp \]
Similarly, \( Q = y^{-1}zq \),

Using these in (1), it becomes

\[
P^2 + Q^2 = 1 \quad \text{...........(2)}
\]

As (2) contains only P and Q explicitly, a solution of the equation will be of the form

\[
Z = aX + bY + c \quad \text{............(3)}
\]

Therefore \( P = a \) and \( Q = b \), obtained from (3) satisfy equation (2)

i.e.

\[
a^2 + b^2 = 1,
\]

\[
\Rightarrow b = \pm \sqrt{1 - a^2}
\]

Therefore the complete solution of (2) is

\[
Z = aX \pm \sqrt{1 - a^2}Y + c
\]

Therefore the complete solution of (1) is

\[
z^2 = ax^2 \pm \sqrt{1 - a^2} y^2 + c
\]

Singular solution does not exist. General solution is found out as usual.

Type D:

Equation of the form \( f\left(\frac{px}{z}, \frac{qy}{z}\right) = 0 \).

By putting \( X = \log x, Y = \log y \) and \( Z = \log z \) the equation reduces to \( f(P, Q) = 0 \),

where \( P = \frac{\partial Z}{\partial X} \) and \( Q = \frac{\partial Z}{\partial Y} \).

Problems:

(1) Solve \( pqxy = z^2 \).

Solution:

Given: \( pqxy = z^2 \) \quad \text{.............}(1)

Rewriting (1),

\[
\left(\frac{px}{z}\right)\left(\frac{qy}{z}\right) = 1. \quad \text{.............}(2)
\]
As (2) contains \( \left( \frac{px}{z} \right) \) and \( \left( \frac{qy}{z} \right) \), we make the substitutions

\[ X = \log x, \quad Y = \log y \quad \text{and} \quad Z = \log z \]

Then

\[ P = \frac{\partial z}{\partial x} = \frac{dz}{dZ} \cdot \frac{\partial Z}{\partial X} \cdot \frac{dX}{dx} = z \cdot P \cdot \frac{1}{x} \]

i.e.

\[ \frac{px}{z} = P \]

Similarly,

\[ \frac{qx}{z} = Q \]

Using these in (2), it becomes

\[ PQ = 1 \quad \ldots \ldots \ldots \ldots \ldots (3) \]

which contains only P and Q explicitly. A solution of (3) is of the form

\[ Z = aX + bY + c \quad \ldots \ldots \ldots \ldots \ldots (4) \]

Therefore \( P = a \) and \( Q = b \), obtained from (4) satisfy equation (3)

i.e.

\[ ab = 1 \quad \text{or} \quad b = \frac{1}{a} \]

Therefore the complete solution of (3) is

\[ Z = aX + \frac{1}{a}Y + c \]

Therefore the complete solution of (1) is

\[ \log z = a \log x + \frac{1}{a} \log y + c \quad \ldots \ldots \ldots \ldots \ldots (5) \]

General solution of (1) is obtained as usual.

**General solution of partial differential equations:**

Partial differential equations, for which the general solution can be obtained directly, can be divided into the following three categories.

1. Equations that can be solved by direct (partial) integration.
2. Lagrange’s linear equation of the first order.
3. Linear partial differential equations of higher order with constant coefficients.

**Equations that can be solved by direct (partial) integration:**

**Problems:**
(1) Solve the equation \( \frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \), if \( u = 0 \) when \( t = 0 \) and \( \frac{\partial u}{\partial t} = 0 \) when \( x = 0 \).

Also show that \( u \to \sin x \), when \( t \to \infty \).

Solution:

Given: \( \frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \), ..................(1)

Integrating (1) partially with respect to \( x \),

\[ \frac{\partial u}{\partial t} = e^{-t} \sin x + f(t) \] ............................(2)

When \( t = 0 \) and \( \frac{\partial u}{\partial t} = 0 \) in (2), we get \( f(t) = 0 \).

Equation (2) becomes \( \frac{\partial u}{\partial t} = e^{-t} \sin x \) ............................(3)

Integrating (3) partially with respect to \( t \), we get

\[ u = -e^{-t} \sin x + g(x) \] ............................(4)

Using the given condition, namely \( u = 0 \) when \( t = 0 \), we get

\[ 0 = -\sin x + g(x) \text{ or } g(x) = \sin x \]

Using this value in (4), the required particular solution of (1) is

\[ u = \sin x \left(1 - e^{-t}\right) \]

Now

\[ \lim_{t \to \infty} u = \sin x \left[ \lim_{t \to \infty} \left(1 - e^{-t}\right) \right] \]

\[ = \sin x \]

i.e. when \( t \to \infty, u \to \sin x \).

(2) Solve the equation \( \frac{\partial z}{\partial x} = 3x - y \) and \( \frac{\partial z}{\partial y} = -x + \cos y \) simultaneously.

Solution: Given

\( \frac{\partial z}{\partial x} = 3x - y \) ..........................(1)

\( \frac{\partial z}{\partial y} = -x + \cos y \) ..........................(2)

Integrating (1) partially with respect to \( x \),
Differentiating (1) partially with respect to $y$,
\[
\frac{\partial z}{\partial y} = -x + f'(y) \tag{4}
\]
Comparing (2) and (4), we get
\[
f'(y) = \cos y \quad f(y) = \sin y + c \tag{5}
\]
Therefore, the required solution is
\[
z = \frac{3x^2}{2} - xy + \sin y + c, \text{ where } c \text{ is an arbitrary constant.}
\]

**Lagrange’s linear equation of the first order:**

A linear partial differential equation of the first order, which is of the form
\[
Pp + Qq = R
\]
where $P, Q, R$ are functions of $x, y, z$ is called Lagrange’s linear equation.

**working rule to solve** $Pp + Qq = R$

(1) To solve $Pp + Qq = R$, we form the corresponding subsidiary simultaneous equations
\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.
\]
(2) Solving these equations, we get two independent solutions $u = a$ and $v = b$.
(3) Then the required general solution is $f(u, v) = 0$ or $u = \phi(v)$ or $v = \psi(u)$.

**Solution of the simultaneous equations**
\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.
\]

**Methods of grouping:**

By grouping any two of three ratios, it may be possible to get an ordinary differential equation containing only two variables, even though $P; Q; R$ are in general, functions of $x, y, z$. By solving this equation, we can get a solution of the simultaneous equations. By this method, we may be able to get two independent solutions, by using different groupings.
Methods of multipliers:

If we can find a set of three quantities l,m,n which may be constants or functions of the variables x,y,z, such that \( lP + mQ + nR = 0 \), then the solution of the simultaneous equation is found out as follows.

\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}
\]

Since \( lP + mQ + nR = 0 \), \( ldx + mdy + ndz = 0 \). If \( ldx + mdy + ndz = 0 \) is an exact differential of some function \( u(x, y, z) \), then we get \( du = 0 \). Integrating this, we get \( u = a \), which is a solution of

\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.
\]

Similarly, if we can find another set of independent multipliers \( l', m', n' \), we can get another independent solution \( v = b \).

Problems:

(1) Solve \( xp + yq = x \).

Solution:

Given: \( xp + yq = x \).

This is of Lagrange’s type of PDE where \( P = x, Q = y, R = x \).

The subsidiary equations are

\[
\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}.
\]

Taking first two members

\[
\frac{dx}{x} = \frac{dy}{y},
\]

Integrating we get

\[
\log x = \log y + \log c_1
\]

i.e.

\[
\frac{x}{y} = c_1
\]

\[
\frac{x}{y} = c_1, \quad \text{ ............(1)}
\]

Taking first and last members

\[
\frac{dx}{x} = \frac{dz}{x},
\]

i.e.

\[
dx = dz.
\]

Integrating we get

\[
x = z = c_2
\]

\[
v = x - z, \quad \text{ .............(2)}
\]
Therefore the solution of the given PDE is \( \phi(u, v) = 0 \) i.e. \( \phi\left(\frac{x}{y}, x - z\right) = 0 \).

(2) Solve the equation \((x - 2z)p + (2z - y)q = y - x\).

Solution:

Given: \( (x - 2z)p + (2z - y)q = y - x \).

This is of Lagrange’s type of PDE where \( P = x - 2z, Q = 2z - y, R = y - x \).

The subsidiary equations are

\[
\frac{dx}{x - 2z} = \frac{dy}{2z - y} = \frac{dz}{y - x}. \quad \text{..........(1)}
\]

Using the multipliers 1,1,1, each ratio in (1)\( = \frac{dx + dy + dz}{0} \).

\[dx + dy + dz = 0.\]

Integrating, we get \( x + y + z = a \) \( \text{..........(2)} \)

Using the multipliers \( y,x,2z \), each ratio in (1)\( = \frac{ydx + xdy + 2zdz}{0} \).

\[dy(2z) + 2zdz = 0.\]

Integrating, we get \( xy + z^2 = b \) \( \text{..........(3)} \)

Therefore the general solution of the given equation is \( f(x + y + z, xy + z^2) = 0 \).

(3) Show that the integral surface of the PDE \( x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2) \).

Which contains the straight line \( x + y = 0, z = 1 \) is \( x^2 + y^2 + 2xyz - 2z + 2 = 0 \).

Solution:

The subsidiary equations of the given Lagrange’s equation are

\[
\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{(x^2 - y^2)}. \quad \text{..........(1)}
\]

Using the multipliers \( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \), each ratio in (1)\( = \frac{1}{x} \frac{dx + 1}{y} \frac{dy + 1}{z} \frac{dz}{0} \).

\[\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0.\]

Integrating, we get \( xyz = a \) \( \text{..........(2)} \)
Using the multipliers $y, x, -1$, each ratio in $(1) = \frac{x \, dx + y \, dy - dz}{0}.$

$$x \, dx + y \, dy - dz = 0.$$ Integrating, we get $x^2 + y^2 - 2z = b$ \hspace{1cm} ............(3)

The required surface has to pass through $x + y = 0$ \hspace{1cm} ............(4)

Using $(4)$ in $(2)$ and $(3)$, we have $-x^2 = a$ \hspace{1cm} ............(5)

$2x^2 - 2 = b$

Eliminating $x$ in $(5)$ we get, $2a + b + 2 = 0$ \hspace{1cm} ............(6)

Substituting for $a$ and $b$ from $(2)$ and $(3)$ in $(6)$, we get $x^2 + y^2 + 2xyz - 2z + 2 = 0$, which is the equation of the required surface.

**Linear P.D.E.S of higher order with constant coefficients:**

The standard form of a homogeneous linear partial differential equation of the $n^{th}$ order with constant coefficients is

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^{n-1} z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^{n-2} z}{\partial x^{n-2} \partial y^2} + \ldots + a_n \frac{\partial^n z}{\partial y^n} = R(x, y) \hspace{1cm} ............(1)$$

where $a$'s are constants.

If we use the operators $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$, we can symbolically write equation $(1)$ as

$$\left(a_0 D^n + a_1 D^{n-1} D' + \ldots + a_n D'^n\right) \, z = R(x, y) \hspace{1cm} ............(2)$$

$$f(D, D') \, z = R(x, y) \hspace{1cm} ............(3)$$

where $f(D, D')$ is a homogeneous polynomial of the $n^{th}$ degree in $D$ and $D'$.

The method of solving $(3)$ is similar to that of solving ordinary linear differential equations with constant coefficients.
The general solution of (3) is of the form $z = (\text{complementary function}) + (\text{particular integral})$, where the complementary function is the R.H.S of the general solution of $f(D, D') z = 0$ and the particular integral is given symbolically by $\frac{1}{f(D, D')} R(x, y)$.

**Complementary function of $f(D, D') z = R(x, y)$:**

C.F of the solution of $f(D, D') z = R(x, y)$ is the R.H.S of the solution of $f(D, D') z = 0$. ..................(1)

In this equation, we put $D = m, D' = 1$, then we get an equation which is called the **auxiliary equation**.

Hence the auxiliary equation of (1) is

$$a_0 m^n + a_1 m^{n-1} + ... + a_n = 0$$ ............................(2)

Let the roots of this equation be $m_1, m_2, ... m_n$.

**Case 1:**

The roots of (2) are real and distinct.

The general solution is given by

$$z = \phi_1 (y + m_1 x) + \phi_2 (y + m_2 x) + ... + \phi_n (y + m_n x)$$

**Case 2:**

Two of the roots of (2) are equal and others are distinct.

The general solution is given by

$$z = \phi_1 (y + m_1 x) + \phi_2 (y + m_1 x) + ... + \phi_n (y + m_n x)$$

**Case 3:**

‘r’ of the roots of (2) are equal and others distinct.

The general solution is given by

$$z = \phi_1 (y + m_1 x) + \phi_2 (y + m_1 x) + ... + x^{r-1} \phi_r (y + m_1 x)$$

**To find particular integral:**

**Rule (1):** If the R.H.S of a given PDE is $f(x, y) = e^{ax+by}$, then

$$P.I = \frac{1}{f(D, D')} e^{ax+by}$$

Put $D = a, D' = b$
\[ P.I = \frac{1}{f(a,b)} e^{ax+by} \quad \text{if} \quad f(a,b) \neq 0 \]

If \( f(a,b) = 0 \), refer to Rule (4).

**Rule (2):** If the R.H.S of a given PDE is \( f(x, y) = \sin(ax + by) \) or \( \cos(ax + by) \), then

\[ P.I = \frac{1}{f(D, D')} \sin(ax + by) \text{ or } \cos(ax + by) \]

Replace \( D^2 = -a^2, D'^2 = -b^2 \) and \( DD' = -ab \) in \( f(D, D') \) provided the denominator is not equal to zero.

If the denominator is zero, refer to Rule (4).

**Rule (3):** If the R.H.S of a given PDE is \( f(x, y) = x^m y^n \), then

\[ P.I = \frac{1}{f(D, D')} x^m y^n = \left\{ f(D, D') \right\}^{-1} (x^m, y^n) \]

Expand \( \left\{ f(D, D') \right\}^{-1} \) by using Binomial Theorem and then operate on \( x^m y^n \).

**Rule (4):** If the R.H.S of a given PDE \( f(x, y) \) is any other function [other than Rule(1),(2) and(3)] resolve \( f(D, D') \) into linear factors say \( (D - m_1 D')(D - m_2 D') \) etc. then the

\[ P.I = \frac{1}{(D - m_1 D')(D - m_2 D')} f(x, y) \]

**Note:** If the denominator is zero in Rule (1) and (2) then apply Rule (4).

**Working rule to find P.I when denominator is zero in Rule (1) and Rule (2).**

If the R.H.S of a given PDE is of the form \( \sin(ax + by) \) or \( \cos(ax + by) \) or \( e^{ax+by} \)

Then

\[ P.I = \frac{1}{(bd - ad')} f(ax + by) = \frac{x^n}{b^n n!} f(ax + by) \]

This rule can be applied only for equal roots.

**Problems:**

(1) Solve \( (D^3 - 3DD'^2 + 2D'^3) \)= \( e^{2x-y} + e^{x+y} \)

**Solution:**

Given: \( (D^3 - 3DD'^2 + 2D'^3) \)= \( e^{2x-y} + e^{x+y} \)
The auxiliary equation is \( m^3 - 3m + 2 = 0 \)
\[ \Rightarrow m = 1, 1, -2 \]

\[ C.F = xf_1(y + x) + f_2(y + x) + f_3(y - 2x) \]

\[ P.I = \frac{1}{D^3 - 3D^2 D' + 2D'^3} \left( e^{2x-y} + e^{x+y} \right) \]
\[ = \frac{1}{(D + 2D')(D - D')^2} e^{2x-y} + \frac{1}{(D - D')(D + 2D')} e^{x+y} \]
\[ = \frac{1}{9} \left( \frac{1}{D + 2D'} e^{2x-y} + \frac{1}{9} \frac{1}{(D - D')} e^{x+y} \right) \]
\[ = \frac{1}{9} \left( xe^{2x-y} + \frac{x^2}{2} e^{x+y} \right) \]

The general solution of the given equation is
\[ z = xf_1(y + x) + f_2(y + x) + f_3(y - 2x) + \frac{x}{9} e^{2x-y} + \frac{x^2}{2} e^{x+y} \]

(2) Solve \( (D^2 + 4DD' - 5D'^2) = xy + \sin(2x + 3y) \)

**Solution:**

Given: \( (D^2 + 4DD' - 5D'^2) = xy + \sin(2x + 3y) \)

The auxiliary equation is \( m^2 + 4m - 5 = 0 \)
\[ \Rightarrow m = 1, -5 \]

\[ C.F = \phi_1(y - 5x) + \phi_2(y + x) \]

\[ (P.I) = \frac{1}{D^2 + 4DD' - 5D'^2} (xy) \]
\[ = \frac{1}{D^2} \left\{ \frac{1}{1 + \frac{D'}{D^2} \left( 4D - 5D' \right)} \right\} (xy) \]
\[ = \frac{1}{D^2} \left\{ \frac{1}{1 - \frac{D'}{D^2} \left( 4D - 5D' \right)} \right\}^{-1} (xy) \]
\[ = \frac{1}{D^2} \left\{ 1 - \frac{D'}{D^2} (4D - 5D') + ... \right\} (xy) \]
Therefore the general solution is

\[ z = \phi_1 (y - 5x) + \phi_2 (y + x) + \frac{1}{6} (x^3 y - \frac{1}{30} x^5 - \frac{1}{17} \sin(2x + 3y)) \]

(3) Solve \( (D^2 - 2DD' + D'^2) y = x^2 y^2 e^{xy} \)

Solution:

Given: \( (D^2 - 2DD' + D'^2) y = x^2 y^2 e^{xy} \)

The auxiliary equation is \( m^2 - m + 1 = 0 \)

\[ \Rightarrow m = 1, 1 \]

C.F. = \( x f_1 (y + x) + f_2 (y + x) \)

P.I. = \( \frac{1}{(D - D')^2} e^{xy} \left( x^2 y^2 \right) \)

\[ = e^{xy} \frac{1}{(D + 1 - (D' + 1))^2} x^2 y^2 \]

\[ = e^{xy} \frac{1}{(D - D')^2} x^2 y^2 \]

\[ = e^{xy} \frac{1}{D^2} \left( 1 - \frac{D'}{D} \right)^2 \left( x^2 y^2 \right) \]

\[ = e^{xy} \frac{1}{D^2} \left( 1 + \frac{2D'}{D} + 3 \frac{D'^2}{D^2} \right) \left( x^2 y^2 \right) \]
Therefore the general solution is

\[ z = x f_1(y + x) + f_2(y + x) + \left( \frac{1}{12} y^2 + \frac{1}{15} xy + \frac{1}{60} x^2 \right) e^{xy} \]
UNIT 1
Part A

(1) Form a partial differential equation by eliminating arbitrary constants a and b from 
\[ z = (x + a)^2 + (y + b)^2 \]
Ans:
Given \( z = (x + a)^2 + (y + b)^2 \) \( \ldots \ldots \) (1)
\[
p = \frac{\partial z}{\partial x} = 2(x + a) \quad \ldots \ldots \) (2)
\[
q = \frac{\partial z}{\partial y} = 2(y + b) \quad \ldots \ldots \) (3)
Substituting (2) & (3) in (1), we get 
\[ z = \frac{p^2}{4} + \frac{q^2}{4} \]

(2) Solve: \( (D^2 - 2DD' + D^2)z = 0 \)
Ans:
Auxiliary equation 
\[ m^2 - 2m + 1 = 0 \]
\[ (m - 1)^2 = 0 \]
\[ m = 1, 1 \]
\[ z = f_1(y + x) + xf_2(y + x) \]

(3) Form a partial differential equation by eliminating the arbitrary constants a and b from the equation 
\( (x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha \).
Ans:
Given: \( (x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha \) \( \ldots \ldots \) (1)
Partially differentiating with respect to ‘x’ and ‘y’ we get 
\[ 2(x - a) = 2zp \cot^2 \alpha \quad \ldots \ldots \) (2)
\[ 2(y - b) = 2zq \cot^2 \alpha \quad \ldots \ldots \) (3)
\[ (2) \Rightarrow \quad x - a = zp \cot^2 \alpha \quad \ldots \ldots \) (4)
\[ (3) \Rightarrow \quad y - b = zq \cot^2 \alpha \quad \ldots \ldots \) (5)
Substituting (4) and (5) in (1) we get 
\[ z^2 p^2 \cot^4 \alpha + z^2 q^2 \cot^4 \alpha = z^2 \cot^2 \alpha \cdot \]
\[ \Rightarrow \quad p^2 + q^2 = \tan^2 \alpha \cdot \]

(4) Find the complete solution of the partial differential equation \( p^2 + q^2 - 4pq = 0 \).
Ans:
Given: 
\[ p^2 + q^2 - 4pq = 0. \quad \ldots \ldots \) (1)
Let us assume that 
\[ z = ax + by + c \quad \ldots \ldots \) (2)
be the solution of (1)
Partially differentiating with respect to ‘x’ and ‘y’ we get
Substituting (3) in (1) we get
\[ a^2 + b^2 - 4ab = 0 \]
From the above equation we get,
\[ a = \frac{4b \pm \sqrt{16b^2 - 4a^2b^2}}{2} \]
\[ a = b \pm b\sqrt{4 - a^2} \] \hspace{1cm} .......... (4)
Substituting (5) in (2) we get
\[ z = b \left( \pm \sqrt{4 - a^2} \right) + by + c \]
(5) Find the PDE of all planes having equal intercepts on the x and y axis.

Ans:
The equation of such plane is
\[ \frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1 \] \hspace{1cm} .......... (1)
Partially differentiating (1) with respect to ‘x’ and ‘y’ we get
\[ \frac{1}{a} + \frac{p}{b} = 0 \Rightarrow p = -\frac{b}{a} \] \hspace{1cm} .......... (2)
\[ \frac{1}{a} + \frac{q}{b} = 0 \Rightarrow q = -\frac{b}{a} \] \hspace{1cm} .......... (3)
From (2) and (3), we get
\[ p = q \]
(6) Find the solution of \( px^2 + qy^2 + 2z^2 \).

Ans:
The S.E is
\[ \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \]
Taking first two members, we get
\[ \frac{dx}{x^2} = \frac{dy}{y^2} \]
Integrating we get
\[ -\frac{1}{x} = -\frac{1}{y} + c_1 \]
i.e
\[ u \left( \frac{1}{y} - \frac{1}{x} \right) = c_1 \]
Taking last two members, we get
\[ \frac{dy}{y^2} = \frac{dz}{z^2} \]
Integrating we get
\[- \frac{1}{y} = - \frac{1}{z} + c_2\]

i.e \[y \left( \frac{1}{z} - \frac{1}{y} \right) = c_2\]

The complete solution is \[
\phi \left( \frac{1}{y} - \frac{1}{z}, \frac{1}{z} - \frac{1}{y} \right) = 0
\]

(7) Find the singular integral of the partial differential equation \( z = px + qy + p^2 - q^2 \).

**Ans:**

The complete integral is \[ z = ax + by + a^2 - b^2 \].

\[
\frac{\partial z}{\partial a} = x + 2a \Rightarrow a = -\frac{x}{2}
\]

\[
\frac{\partial z}{\partial b} = y - 2b \Rightarrow b = \frac{y}{2}
\]

Therefore

\[
z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4} = -\frac{x^2}{4} + \frac{y^2}{4}
\]

\[\Rightarrow y^2 - x^2 = 4z\]

(8) Solve: \( p^2 + q^2 = m^2 \).

**Ans:**

*Given* \( p^2 + q^2 = m^2 \) \[ \cdots \cdots \text{(1)} \]

Let us assume that \[ z = ax + by + c \] \[ \cdots \cdots \text{(2)} \]

be the solution of (1)

Partially differentiating with respect to ‘x’ and ‘y’ we get

\[
p = \frac{\partial z}{\partial x} = a
\]

\[
q = \frac{\partial z}{\partial y} = b
\]

Substituting (3) in (1) we get

\[ a^2 + b^2 = m^2 \]

This is the required solution.

(9) Form a partial differential equation by eliminating the arbitrary constants a and b from \( z = ax^n + by^n \).

**Ans:**

*Given* \( z = ax^n + by^n \). \[ \cdots \cdots \text{(1)} \]

Partially differentiating with respect to ‘x’ and ‘y’ we get
\[ p = \frac{\partial z}{\partial x} = a.nx^{n-1} \quad \Rightarrow \quad a = \frac{p}{nx^{n-1}} \quad \text{......... (2)} \]
\[ q = \frac{\partial z}{\partial y} = b.ny^{n-1} \quad \Rightarrow \quad b = \frac{q}{ny^{n-1}} \]

Substituting (2) in (1) we get
\[ z = \frac{p}{nx^{n-1}} x^n + \frac{q}{ny^{n-1}} y^n \]
\[ z = \frac{1}{n} (px + qy) \]

This is the required PDE.

(10) Solve:
\[ \left( D^3 + D^2 D' + DD'^2 + D'^3 \right) \varepsilon = 0. \]

\textbf{Ans:}

Auxiliary equation
\[ m^3 + m^2 + m + 1 = 0 \]
\[ (m+1)^3 = 0 \]
\[ m = -1, -1, -1 \]
\[ z = f_1(y-x) + xf_2(y-x) + x^2 f_3(y-x) \]

(11) Form a partial differential equation by eliminating the arbitrary constants \( a \) and \( b \) from
\[ z = \left( x^2 + a^2 \right) y^2 + b^2 \]

\textbf{Ans:}

Given \( z = \left( x^2 + a^2 \right) y^2 + b^2 \) \quad \text{......... (1)}
\[ p = \frac{\partial z}{\partial x} = 2x(y^2 + b^2) \quad \Rightarrow \quad y^2 + b^2 = \frac{p}{2x} \quad \text{......... (2)} \]
\[ q = \frac{\partial z}{\partial y} = 2y(x^2 + a^2) \quad \Rightarrow \quad x^2 + a^2 = \frac{q}{2y} \quad \text{......... (3)} \]

Substituting (2) & (3) in (1), we get
\[ z = \frac{q}{2y} \cdot \frac{p}{2x} \]
\[ pq = 4xyz \]

(12) Solve:
\[ \left( D^2 - DD' + D' - 1 \right) \varepsilon = 0 \]

\textbf{Ans:}

The given equation can be written as
\[ (D-1)(D - D' + 1) \varepsilon = 0 \]

We know that the C.F corresponding to the factors
\[ (D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2) \varepsilon = 0 \]
\[ z = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x) \]

In our problem
\[ \alpha_1 = 1, \alpha_2 = -1, m_1 = 0, m_2 = 1 \]
\[ C.F = e^x f_1(y + x) + e^{-x} f_2(y + x) \]
\[ z = e^x f_1(y + x) + e^{-x} f_2(y + x) \]
(13) Form a partial differential equation by eliminate the arbitrary function \( f \) from

\[
z = f\left(\frac{xy}{z}\right)
\]

**Ans:**

Given: \( z = f\left(\frac{xy}{z}\right) \).

\[
p = f'\left(\frac{xy}{z}\right) \cdot \frac{zy - xy \cdot p}{z^2}
\]

\[
q = f''\left(\frac{xy}{z}\right) \cdot \frac{zx - xy \cdot q}{z^2}
\]

From (1), we get

\[
f''\left(\frac{xy}{z}\right) = \frac{pz^2}{zy - xy \cdot p}
\]

Substituting (3) in (2), we get

\[
p = \frac{pz^2}{zy - xy \cdot p}
\]

(14) Solve:

\[
\left(D^3 + 2D^2D' - DD'^2 - 2D'^3\right) = 0.
\]

**Ans:** Auxiliary equation

\[
m^3 + 2m^2 - m - 2 = 0
\]

\[
m = 1, -1, -2
\]

**Solution is** \( z = f_1(y + x) + f_2(y - x) + f_3(y - 2x) \)

(15) Obtain partial differential equation by eliminating arbitrary constants \( a \) and \( b \) from

\[
(x + a)^2 + (y + b)^2 + z^2 = 1
\]

**Ans:**

Given \((x + a)^2 + (y + b)^2 + z^2 = 1\) \hspace{1cm} (1)

\[
2(x + a) + 2zp = 0 \quad \Rightarrow \quad (x + a) = -zp \hspace{1cm} (2)
\]

\[
2(y + b) + 2zq = 0 \quad \Rightarrow \quad (y + b) = -zq \hspace{1cm} (3)
\]

Substituting (2) & (3) in (1), we get

\[
z^2 p^2 + z^2 q^2 + z^2 = 1
\]

\[
p^2 + q^2 + 1 = \frac{1}{z^2}
\]

(16) Find the general solution of

\[
4 \frac{\partial^2 z}{\partial y^2} - 12 \frac{\partial^2 z}{\partial x \partial y} + 9 \frac{\partial^2 z}{\partial^2 y} = 0.
\]

**Ans:** Auxiliary equation is

\[
4m^2 - 12m + 9 = 0
\]

\[
m = \frac{12 \pm \sqrt{144 - 144}}{8}
\]
General solution is
\[ z = f_1 \left( y + \frac{3}{2} x \right) + xf_2 \left( y + \frac{3}{2} x \right) \]

(17) Find the complete integral of
\[ p + q = pq, \quad \text{where} \quad p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}. \]

**Ans:** Let us assume that
\[ z = ax + by + c \]
be the solution of the given equation.

Partially differentiating with respect to ‘x’ and ‘y’ we get
\[ p = \frac{\partial z}{\partial x} = a \]
\[ q = \frac{\partial z}{\partial y} = b \]

Substituting (2) in (1) we get
\[ a + b = ab \Rightarrow b = \frac{a}{a - 1} \]

Substituting the above in (1) we get
\[ z = ax + \left( \frac{a}{a - 1} \right)y + c \]

This gives the complete integral.

(18) Solve:
\[ \left( D^3 - 3DD'^2 + 2D'' \right) z = 0 \]

**Ans:**
Auxiliary equation
\[ m^3 - 3m + 2 = 0 \]
\[ m = 1, 1, -2 \]

Solution is
\[ z = f_1 (y + x) + xf_2 (y + x) + f_3 (y - 2x) \]

(19) Find the PDE of the family of spheres having their centers on the line x = y = z.

**Ans:** The equation of such sphere is
\[ (x - a)^2 + (y - a)^2 + (z - a)^2 = r^2 \]

Partially differentiating with respect to ‘x’ and ‘y’ we get
\[ 2(x - a) + 2(z - a)p = 0 \]
\[ 2(y - a) + 2(z - a)q = 0 \]

From (1),
\[ a = \frac{x + zp}{1 + p} \]

From (2),
\[ a = \frac{y + zq}{1 + q} \]
From (3) and (4), we get
\[
\frac{x + zp}{1 + p} = \frac{y + zq}{1 + q}
\]
This is the required PDE.

(20) Solve:
\[
\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^2 z}{\partial x^2 \partial y} - 4 \frac{\partial^2 z}{\partial x \partial y^2} + 8 \frac{\partial^3 z}{\partial y^3} = 0.
\]

**Ans:**
Auxiliary equation
\[m^3 - 2m^2 - 4m + 8 = 0\]
\[m = 2, 2, -2\]
Solution is
\[z = f_1(y + 2x) + xf_2(y + 2x) + f_3(y - 2x)\]

**Part B**

(1)(i) Form a partial differential equation by eliminating arbitrary functions from
\[z = xf(2x + y) + g(2x + y)\]
(ii) Solve: \[p^2 y(l + x^2) = qx^2\]

(2)(i) Solve:
\[x(e^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)\]
(ii) Solve:
\[\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^2 z}{\partial x^2 \partial y} - 4 \frac{\partial z}{\partial x \partial y^2} = e^{x+y} + 4 \sin(x + y)\]

(3)(i) Solve:
\[(x^2 - y^2 - z^2)p + 2xyq - 2xz = 0\]
(ii) Solve:
\[\left(D^2 - DD' - 2D'^2\right)\frac{z}{x} - 2\frac{z}{x} = 3y + e^{3x+y}\]

(4)(i) Solve:
\[z^2(p^2 + q^2) = x^2 + y^2\]
(ii) Solve:
\[\left(D^2 + 3DD' - 4D'^2\right)\frac{k}{x} = \sin y\]

(5)(i) Solve:
\[(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)\]
(ii) Solve:
\[\left(D^2 - DD' - 20D'^2\right)\frac{k}{x} = e^{5x+y} + \sin(4x - y)\]

(6)(i) Solve:
\[z = p^2 + q^2\]
(ii) Solve:
\[\left(D^2 + DD' - 6D'^2\right)\frac{k}{x} = x^2 + e^{3x+y}\]

(7)(i) Solve:
\[(y^2 + z^2)p - xyq + xz = 0\]
(ii) Solve:
\[\left(D^2 - 6DD' + 5D'^2\right)\frac{k}{x} = xy + e^x \sinh y\]

(8)(i) Solve:
\[p(l - q^2) = q(l - z)\]
(ii) Solve:
\[\left(D^2 - 4DD' + 4D'^2\right)\frac{k}{x} = xy + e^{2x+y}\]
(9)(i) Solve: \( (x^2 - yz)p + (y^2 - zx)q = z^2 - xy \)
(ii) Solve: \( x(y - z)p + y(z - x)q = z(x - y) \)

(10)(i) Solve: \( p^2 + x^2 y^2 q^2 = x^2 z^2 \)
(ii) Solve: \( (D^2 - D'^2 - 3D + 3D')e = xy \)

(11)(i) Form the partial differential equation by eliminating \( f \) and \( \phi \) from 
\[ z = f(y) + \phi(x + y + z) \]
(ii) Solve: \( (D'^2 - 2DD')e = x^3 y + e^{2x} \)

(12)(i) Find the complete integral of \( p + q = x + y \)
(ii) Solve: \( y^2 p - xy q = x(z - 2y) \)

(13)(i) Solve: \( (3z - 4y)p + (4x - 2z)q = 2y - 3x \)
(ii) Solve: \( (D^2 - 2DD' - D'^2 + 3D + 3D' + 2)e = (e^{3x} + 2e^{-2y}) \)
(iii) Solve: \( (D^2 + 4DD' - 5D'^2)e = \sin(x - 2y) + e^{2x-y} \)

(14)(i) Solve: \( z^2 = 1 + p^2 + q^2 \)
(ii) Solve: \( (y - z)p - (2x + y)q = 2x + z \)

(15)(i) Form a partial differential equation by eliminating arbitrary functions \( f \) and \( g \) in 
\[ z = x^2 f(y) + y^2 g(x) \]
(ii) Solve: \( (D^2 - DD' - 20D'^2)e = \sin(4x - y) + e^{5x+y} \)

(16)(i) Form a partial differential equation by eliminating arbitrary functions \( f \) and \( g \) in 
\[ z = f(x^2 + 2y) + g(x^3 - 2y) \]
(ii) Solve: \( (y - xy)p + (yz - x)q = (x + y)(x - y) \)

(17)(i) Find the singular solution of \( z = px + qy + \sqrt{p^2 + q^2 + 1} \)
(ii) Solve: \( (D^2 - DD' - 30D'^2)e = xy + e^{6x+y} \)

(18) (i) Solve: \( (D^3 - 7DD'^2 - 6D'^3)e = \sin(x + 2y) + e^{2x+y} \)
(ii) Find the singular integral of a partial differential equation \( z = px + qy + p^2 - q^2 \)

(19)(i) Solve: \( (4D^2 - 4DD' + D'^2)e = 16 \log(x + 2y) \)
(ii) Form a partial differential equation by eliminating arbitrary functions \( f \) from 
\[ f(z - xy, x^2 + y^2) = 0 \]

(20)(i) Solve: \( (D^2 + D'^2)e = \sin 2x \sin 3y + 2 \sin^2(x + y) \)
(ii) Solve: \( p^2 + x^2 y^2 q^2 = x^2 z^2 \)