CHAPTER 3

APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

In many physical and engineering problems, we always seek a solution of the differential equations, whether it is ordinary or partial, which satisfies some specified conditions called the boundary conditions.

Any differential equations together with these boundary conditions is called boundary value problem.

In this chapter we shall study some of the most important partial differential equations occurring in engineering applications.

One of the most fundamental common phenomena that are found in nature is the phenomena of wave motion. When a stone is dropped into a pound, the surface of water is disturbed and waves of displacement travel rapidly outward. When a bell or tuning fork is struck, sound waves are propagated from the source of sound.

Whatever is the nature of wave phenomena, whether it is the displacement of a tightly stretched string, the deflection of a stretched membrane, the propagation of currents and potentials along an electrical transmission line, these entities are governed by a partial differential equation, known as the Wave Equation.

Variable Separable Solution of the Wave Equation \( \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \)

Let \( y(x,t) = X(x)T(t) \)

be the solution of the equation

\( \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \)

Where \( X(x) \) is a function of \( x \) alone and \( T(t) \) is a function of \( t \) alone. Then

\( \frac{\partial^2 y}{\partial t^2} = XT'' \) and \( \frac{\partial^2 y}{\partial x^2} = X''T \), Where \( T'' = \frac{d^2 T}{dt^2} \) and \( X'' = \frac{d^2 x}{dx^2} \) Satisfy equation (2)

i.e., \( XT'' = a^2 X''T \)

\( \frac{X''}{X} = \frac{T''}{a^2 T} \)

\\[ \text{The L.H.S of (3) is a function of } x \text{ alone and the R.H.S is a function of } t \text{ alone. They are equal for all values of the independent variable } x \text{ and } t. \text{ This is possible only if each is a constant.} \]

\( \frac{X''}{X} = \frac{T''}{a^2 T} = k \), Where k is a constant.
The nature of the solution of (4) and (5) depends on the nature of values of $k$. Hence the following three cases arise.

**Case 1:**
$k$ is positive. Let $k = p^2$
Then equation (4) and (5) become
\[(D^2 - p^2)X = 0\]
and
\[(D'^2 - p^2 a^2)T = 0\]
Where
\[D \equiv \frac{d}{dx} \quad \text{and} \quad D' \equiv \frac{d}{dt}\]
The solutions of these equations are
\[X = Ae^{px} + Be^{-px}\]
and
\[T = Ce^{pat} + De^{-pat}\]

**Case 2:**
k is negative. Let $k = -p^2$
Then equation (4) and (5) become
\[(D^2 + p^2)X = 0\]
and
\[(D'^2 + p^2 a^2)T = 0\]
The solutions of these equations are
\[X = A \cos px + B \sin px\]
and
\[T = C \cos pat + D \sin pat\]

**Case 3:**
k = 0.
Then equation (4) and (5) become
\[\frac{d^2 X}{dx^2} = 0\]
and
\[\frac{d^2 T}{dt^2} = 0\]
The solutions of these equations are
\[X = Ax + B\]
and
\[T = Ct + D\]
Since $y(x,t) = X.T$ is the solution of the wave equation, the three mathematically possible solutions of the wave equations are
Problems:
(1) A uniform string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form of the curve

(i) \( y = k \sin^3 \left( \frac{\pi x}{l} \right) \) and

(ii) \( y = k x (l - x) \)

and then releasing it from this position at time \( t=0 \). Find the displacement of the point of the string at a distance \( x \) from one end at time \( t \).

Solution:

The displacement \( y(x,t) \) of the point of the string at a distance \( x \) from the left end 0 at time \( t \) is given by the equation (fig.1).

\[
\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}
\]  \( \quad \ldots \ldots \ldots (1) \)

Since the ends of the string \( x=0 \) and \( x=l \) are fixed, they do not undergo any displacement at any time.

Hence \( y(x,t) = 0, \quad for \quad t \geq 0 \) \( \ldots \ldots \ldots (2) \)

and \( y(l,t) = 0, \quad for \quad t \geq 0 \) \( \ldots \ldots \ldots (3) \)

Since the string is released from rest initially, that is, at \( t=0 \), the initial velocity of every point of the string in the y-direction is zero.

Hence \( \frac{\partial y}{\partial t}(x,0) = 0, \quad for \quad 0 \leq x \leq l \) \( \ldots \ldots \ldots (4) \)

Since the string is initially displaced in to the form of the curve \( y = f(x) \), the coordinates
\{x, y(x,0)\} satisfy the equation \( y = f(x) \), where \( y(x,0) \) is the initial displacement of the point ‘x’ in the y-direction.

Hence \( y(x,0) = f(x) \) for \( 0 \leq x \leq l \) ........... (5)

Where in (i) and in (ii). Conditions (2),(3),(4) and (5) are collectively called boundary conditions of the problem. We have to get the solution of equation (1), the appropriate solution, consistent with the vibration of the string is

\[ y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \] ........... (6)

Where A, B, C, D and p are arbitrary constants that are to be found out by using the boundary conditions.

Using boundary conditions (2) in (6), we have

\[ A(C \cos pat + D \sin pat) = 0 \text{ for all } t \geq 0 \]

\[ A = 0 \]

Using boundary conditions (3) in (6), we have

\[ B \sin pl(C \cos pat + D \sin pat) = 0 \text{ for all } t \geq 0 \]

\[ B \sin pl = 0 \]

Either \( B = 0 \) or \( \sin pl = 0 \).

If \( B = 0 \), the solution becomes \( y(x,t) = 0 \), which is meaningless.

\[ \sin pl = 0 \]

\[ pl = n\pi \]

\[ p = \frac{n\pi}{l} \]

Where \( n = 0,1,2,\ldots \infty \)

Differentiating both sides of (6) partially with respect to \( t \), we have

\[ \frac{\partial y}{\partial t}(x,t) = (B \sin px)pat - (C \sin pat + D \cos pat) \] ........... (7)

Where \( p = \frac{n\pi}{l} \)

Using boundary conditions (4) in (7), we have

\[ B \sin px \cdot pa \cdot D = 0 \text{ for } 0 \leq x \leq l \]

As \( B \neq 0 \) and \( p \neq 0 \), we get \( D = 0 \)

Using these values of A, p, D in (6), the solution reduces to

\[ y(x,t) = BC \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}, \text{ where } n = 1,2,3,\ldots \infty \]

Taking \( BC = k \), Eq.(1) has infinitely many solutions given below.

\[ y(x,t) = k \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} \]

\[ y(x,t) = k \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l} \]

\[ y(x,t) = k \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l}, \text{ etc.} \]
Since Eq. (1) is linear, a linear combination of the R.H.S members of all the above solutions is the general solution of Eq. (1). Thus the most general solution of Eq. (1) is

\[ y(x,t) = \sum_{n=1}^{\infty} (c_n k) \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \]

or

\[ y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \]

\[ \text{.........} (8) \]

Where \( \lambda_n \) is yet to be found out.

Using boundary conditions (5) in (8), we have

\[ \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = f(x) \text{ for } 0 \leq x \leq l \]

\[ \text{.........} (9) \]

If we can express \( f(x) \) in a series comparable with the L.H.S. series of (9), we can get the values of \( \lambda_n \).

(i) \( f(x) = k \sin^{3} \frac{\pi x}{l} \)

\[ = \frac{k}{4} \left( 3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right) \]

Using this form of \( f(x) \) in (9) and comparing like terms, we get

\( \lambda_1 = \frac{3k}{4}, \lambda_2 = -\frac{k}{4}, \lambda_3 = 0 = \lambda_4 = \ldots \)

Using these values in (8), the required solution is

\[ y(x,t) = \frac{3k}{4} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} - \frac{k}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l} \]

(ii) \( y = kx(l - x) \)

If we expand \( f(x) \) as Fourier half-range sine series in \((0, l)\), that is in the form

\[ \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \]

it is comparable with the L.H.S series of (9).

Thus

\[ \lambda_n = b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx, \text{ by Euler’s formula} \]

\[ = \frac{2k}{l} \int_0^l \left( x - x^2 \right) \sin \frac{n\pi x}{l} \, dx \]

\[ = \frac{2k}{l} \left\{ \left( x - x^2 \right) \left( \frac{\cos \frac{n\pi x}{l}}{n\pi l} \right) \right\} \bigg|_0^l - \left( l - 2x \right) \left( \frac{\sin \frac{n\pi x}{l}}{n^2 \pi^2 l^2} \right) \bigg|_0^l + (-2) \left( \frac{\cos \frac{n\pi x}{l}}{n^3 \pi^3 l^3} \right) \bigg|_0^l \]

\[ = \frac{4kl^2}{n^3 \pi^3} \left\{ (-1)^n \right\} \]
Using this value of $\lambda_n$ in (8), the required solution is

$$y(x,t) = \frac{8k^2l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{l} \cos \frac{(2n-1)\pi at}{l}$$

(2) Solve the one dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ in $-l \leq x \leq l, t \geq 0$,

given that $y(-l,t) = 0, y(l,t) = 0, \frac{\partial y}{\partial t}(x,0) = 0$ and $y(x,0) = \frac{b}{l} (l - |x|)$

**Solution:**

Shifting the origin to the point $(-l,0)$, we get $x = X - l$ and $y = Y$, Where $(X,Y)$ are the coordinates of the point $(x, y)$ with reference to the new system of coordinate axes. The differential equation in the new system is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad 0 \leq X \leq 2l, t \geq 0, \quad \ldots \ldots \text{(1)}$$

The boundary conditions become

$$Y(0,t) = 0 \quad \ldots \ldots \text{(2)}$$
$$Y(2l,t) = 0 \quad \ldots \ldots \text{(3)}$$

for all $t \geq 0$,

$$\frac{\partial Y}{\partial t}(X,0) = 0 \quad \ldots \ldots \text{(4)}$$

and

$$Y(X,0) = \begin{cases} \frac{b}{l} X, & \text{in } 0 \leq X \leq l \\ \frac{b}{l} (2l - X), & \text{in } l \leq X \leq 2l \end{cases} \quad \ldots \ldots \text{(5)}$$

Since the last boundary condition in the old system is

$$y(x,0) = \begin{cases} \frac{b}{l} (1 + x), & \text{in } -l \leq x \leq 0 \\ \frac{b}{l} (1 - x), & \text{in } 0 \leq x \leq l \end{cases}$$

The required solution of equation (1) is

$$Y(X,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{\pi x}{2l} \sin \frac{n\pi X}{2l} \cos \frac{n\pi at}{2l}$$

Since $\sin \frac{n\pi}{2}$, When $n$ is an even integer, the solution can be rewritten as

$$Y(X,t) = \frac{8b}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \sin \frac{\pi x}{2l} \sin \frac{n\pi X}{2l} \cos \frac{n\pi at}{2l} \quad \text{Where } 0 \leq X \leq 2l, t \geq 0.$$

Changing over to the old system of coordinates, the solution becomes
Now since \( n \) is odd.

The required solution is

\[
\begin{align*}
\sin \frac{n\pi}{2l} (x + l) &= \sin \left( \frac{n\pi}{2} + \frac{n\pi x}{2l} \right) \\
&= \sin \frac{n\pi}{2} \cos \frac{n\pi x}{2l} + \cos \frac{n\pi}{2} \sin \frac{n\pi x}{2l} \\
&= \sin \frac{n\pi}{2} \cos \frac{n\pi x}{2l}, \text{ since } n \text{ is odd.}
\end{align*}
\]

The required solution is

\[
y(x, t) = \frac{8b}{\pi^2} \sum_{n=1,3,5,\ldots}^\infty \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l}
\]

\[
y(x, t) = \frac{8b}{\pi^2} \sum_{n=1,3,5,\ldots}^\infty \frac{1}{n^2} \cos \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l}
\]

Where \(-l \leq x \leq l\) and \( t \geq 0 \).

(3) A tightly stretched strings with fixed end points \( x=0 \) and \( x=50 \) is initially at rest in its equilibrium position. If it is said to vibrate by giving each point a velocity

(i) \( v = v_0 \sin \frac{\pi x}{50} \) and

(ii) \( v = v_0 \sin \frac{\pi x}{50} \cos \frac{2\pi x}{50} \).

Find the displacement of any point of the string at any subsequent time.

**Solution:**

The displacement \( y(x, t) \) of any point 'x' of the string at any time 't' is given by

\[
\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}
\]

\[
\ldots (1)
\]

We have to solve equation (1) satisfying the following boundary conditions.

\[
y(0, t) = 0, \text{ for } t \geq 0 \quad \ldots (2)
\]

\[
y(50, t) = 0, \text{ for } t \geq 0 \quad \ldots (3)
\]

\[
y(x, 0) = 0, \text{ for } 0 \leq x \leq 50 \quad \ldots (4)
\]

Since the string is in its equilibrium position initially and so the initial displacement of every point of the string is zero.

\[
\frac{\partial y}{\partial t}(x, 0) = f(x) \quad \text{for } 0 \leq x \leq 50 \quad \ldots (5)
\]

where \( f(x) = v_0 \sin \frac{\pi x}{50} \) For (i) and

\[
f(x) = v_0 \sin \frac{\pi x}{50} \cos \frac{2\pi x}{50} \quad \text{For (ii)}
\]

The suitable solution of Eq (1), consistent with the vibration of the string, is

\[
y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \quad \ldots (6)
\]

Using boundary conditions (2) in (6), we have
Using boundary conditions (3) in (6), we have

\[ A(C \cos pt + D \sin pt) = 0 \quad \text{for all } t \geq 0 \]

\[ A = 0 \]

Using boundary conditions (3) in (6), we have

\[ B \sin 50p(C \cos pt + D \sin pt) = 0 \quad \text{for all } t \geq 0 \]

Either \( B = 0 \) or \( \sin 50p = 0 \)

If we assume that \( B = 0 \), we get a trivial solution.

\[ \sin 50p = 0 \]

\[ 50p = n\pi \]

\[ p = \frac{n\pi}{50} \]

Where \( n = 0, 1, 2, \ldots \infty \)

Using boundary conditions (4) in (6), we have

\[ B \sin px C = 0 \quad \text{for} \quad 0 \leq x \leq 50 \]

As \( B \neq 0 \), we get \( C = 0 \)

Using these values of \( A, p, C \) in (6), the solution reduces to

\[ y(x, t) = k \sin \frac{n\pi x}{50} \sin \frac{n\pi t}{50} \]

where \( k = BD \) and \( n = 1, 2, 3, \ldots \infty \)

The most general solution of Eq.(1) is

\[ y(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{50} \sin \frac{n\pi t}{50} \]

Differentiating both sides of (8) partially with respect to \( t \), we have

\[ \frac{\partial y}{\partial t} (x, t) = \sum_{n=1}^{\infty} \left( \frac{n\pi}{50} \lambda_n \right) \sin \frac{n\pi x}{50} \cos \frac{n\pi t}{50} \]

Using boundary condition (5) in (9), we have

\[ \sum_{n=1}^{\infty} \left( \frac{n\pi}{50} \lambda_n \right) \sin \frac{n\pi x}{50} = v. \quad \text{Since} \quad v = \frac{\partial y}{\partial t} (x, 0) \]

(i) \[ v = v_0 \sin \frac{3\pi x}{50} \]

and

\[ = v_0 \left( 3 \sin \frac{\pi x}{50} - \sin \frac{3\pi x}{50} \right) \]

\[ \sum_{n=1}^{\infty} \left( \frac{n\pi}{50} \lambda_n \right) \sin \frac{n\pi x}{50} = v_0 \left( 3 \sin \frac{\pi x}{50} - \sin \frac{3\pi x}{50} \right) \]

Comparing like terms, we get

\[ \lambda_1 = \frac{3v_0}{4}, \quad \lambda_3 = \frac{3v_0}{4} \quad \text{and} \quad \sum_{n=1}^{\infty} \left( \frac{n\pi}{50} \lambda_n \right) = 0, \quad \text{for} \quad n = 2, 4, 5, 6, \ldots \infty \]

\[ \lambda_1 = \frac{75v_0}{2\pi a}, \quad \lambda_3 = \frac{25v_0}{6\pi a} \quad \text{and} \quad \lambda_2 = 0 = \lambda_4 = \lambda_5 = \ldots \]

Using these values in (8), the required solution is
Comparing like terms, we get

\[
y = v_0 \sin \frac{\pi x}{50} \cos \frac{2\pi x}{50}
\]

\[= \frac{v_0}{2} \left( \sin \frac{3\pi x}{50} - \sin \frac{\pi x}{50} \right)
\]

\[
\sum_{n=1}^{\infty} \left( \frac{n\pi a}{50} \lambda_n \right) \sin \frac{n\pi x}{50} = \frac{v_0}{2} \left( \sin \frac{3\pi x}{50} - \sin \frac{\pi x}{50} \right).
\]

Using these values in (8), the required solution is

\[
y(x, t) = -\frac{25v_0}{\pi a} \sin \frac{\pi x}{50} \sin \frac{\pi a t}{50} + \frac{25v_0}{3\pi a} \sin \frac{3\pi x}{50} \sin \frac{3\pi a t}{50}
\]

(4) A taut string of length \(2l\), fastened at both ends, is disturbed from its position of equilibrium by imparting to each of its points an initial velocity of magnitude \(k(2lx - x^2)\). Find the displacement function \(y(x, t)\).

Solution:
The displacement \(y(x, t)\) of any point ‘x’ of the string at any time ‘t’ is given by

\[
\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}
\]

We have to solve equation (1) satisfying the following boundary conditions.

\[
y(0, t) = 0, \quad \text{for} \quad t \geq 0
\]

\[
y(2l, t) = 0, \quad \text{for} \quad t \geq 0
\]

\[
y(x, 0) = 0, \quad \text{for} \quad 0 \leq x \leq 2l
\]

\[
\frac{\partial y}{\partial t}(x, 0) = k(2lx - x^2) \quad \text{for} \quad 0 \leq x \leq 2l
\]

The suitable solution of Eq (1), consistent with the vibration of the string, is

\[
y(x, t) = \left( A \cos px + B \sin px \right) \left( C \cos pat + D \sin pat \right)
\]

Using boundary conditions (2) in (6), we have

\[A \left( C \cos pat + D \sin pat \right) = 0 \quad \text{for all } t \geq 0\]

\[A = 0\]

Using boundary conditions (3) in (6), we have

\[B \sin 2lp \left( C \cos pat + D \sin pat \right) = 0 \quad \text{for all } t \geq 0\]

Either \[B = 0 \text{ or } \sin 2lp = 0\]

If we assume that \(B = 0\), we get a trivial solution.
Where \( n = 0, 1, 2 \cdots \infty \)

Using boundary conditions (4) in (6), we have

\[
B \sin px. C = 0 \quad \text{for} \quad 0 \leq x \leq 2l
\]

As \( B \neq 0 \), we get \( C = 0 \)

Using these values of \( A, p, C \) in (6), the solution reduces to

\[
y(x, t) = k \sin \frac{n \pi x}{2l} \sin \frac{n \pi a t}{2l}, \quad \text{where} \quad n = 0, 1, 2, 3 \cdots \infty
\]

The most general solution of Eq.(1) is

\[
y(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n \pi x}{2l} \cos \frac{n \pi a t}{2l}, \quad \text{......... (8)}
\]

Differentiating both sides of (8) partially with respect to \( t \), we have

\[
\frac{\partial y}{\partial t} (x, t) = \sum_{n=1}^{\infty} \left( \frac{n \pi a}{2l} \lambda_n \right) \sin \frac{n \pi x}{2l} \cos \frac{n \pi a t}{2l} \quad \text{......... (9)}
\]

Using boundary condition (5) in (9), we have

\[
\sum_{n=1}^{\infty} \left( \frac{n \pi a}{2l} \lambda_n \right) \sin \frac{n \pi x}{2l} = k \left( 2lx - x^2 \right) \quad \text{for} \quad 0 \leq x \leq 2l
\]

\[
= \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{2l}
\]

Which is Fourier half-range sine series of \( k \left( 2lx - x^2 \right) \) in \((0,2l)\).

Comparing like terms, we get

\[
\frac{n \pi a}{2l} \lambda_n = b_n = \frac{2}{2l} \int_0^{2l} f(x) \sin \frac{n \pi x}{l} \, dx, \text{by Euler's formula}
\]

\[
= \frac{2}{2l} \int_0^{2l} k \left( 2lx - x^2 \right) \sin \frac{n \pi x}{2l} \, dx
\]

\[
= \frac{2k}{n \pi a} \left\{ \left( 2lx - x^2 \right) \left( -\frac{\cos \frac{n \pi x}{2l}}{\frac{n \pi}{2l}} \right) - \left( 2l - 2x \right) \left( -\frac{\sin \frac{n \pi x}{2l}}{\frac{n \pi}{2l}} \right) + \left( \frac{\cos \frac{n \pi x}{2l}}{\frac{n \pi}{2l}} \right) \right\}^{2l}_0
\]

\[
= \frac{32kl^3}{n^4 \pi^4 a} \left\{ \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-(-1)^n}{n^4 \pi^4 a}, & \text{if } n \text{ is odd} \end{cases} \right.
\]
Using this value of $\lambda_n$ in (8), the required solution is

$$y(x,t) = \frac{64k^3}{\pi^4a} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \left( \frac{(2n-1)\pi x}{2l} \right) \cos \left( \frac{(2n-1)\pi a t}{2l} \right)$$

**ONE DIMENSIONAL HEAT FLOW**

**VARIABLE SEPARABLE SOLUTIONS OF THE HEAT EQUATION**

The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Let $u(x,t) = X(x)T(t) \tag{2}$ be a solution of Eq.(1), where $X(x)$ is a function of $x$ alone and $T(t)$ is a function of $t$ alone. Then

$$\frac{\partial u}{\partial t} = XT'$$

and

$$\frac{\partial^2 u}{\partial x^2} = X''T,$$ where $T' = \frac{dT}{dt}$ and $X'' = \frac{d^2X}{dx^2}$, satisfy Eq.(1).

i.e.,

$$XT' = \alpha^2 X''T \tag{3}$$

The L.H.S. of (3) is a function of $x$ alone and the R.H.S is a function of $t$ alone.

They are equal for all values of independent variables $x$ and $t$. This is possible only if each is a constant.

$$\therefore \quad \frac{X''}{X} = \frac{T'}{\alpha^2 T} = k,$$ where $k$ is a constant.

$$\therefore \quad X'' - kX = 0 \tag{4}$$

and

$$T' - k\alpha^2 T = 0 \tag{5} \tag{4}$$

The nature of the solutions of (4) and (5) depends on the nature of the values of $k$. Hence the following three cases come into being.

**Case 1**: $k$ is positive. Let $k = p^2$.

Then equations (4) and (5) become

$$(D^2 - p^2)X = 0 \quad \text{and} \quad (D' - p^2\alpha^2)T = 0,$$ where

$$D = \frac{d}{dx} \quad \text{and} \quad D' = \frac{d}{dt}.$$

The solutions of these equations are

$$X = C_1 e^{px} + C_2 e^{-px} \quad \text{and} \quad T = C_3 e^{p\alpha^2 t}$$

**Case 2**: $k$ is negative. Let $k = -p^2$.

Then equations (4) and (5) become

$$(D^2 + p^2)X = 0 \quad \text{and} \quad (D' + p^2\alpha^2)T = 0,$$

The solutions of these equations are

$$X = C_1 \cos px + C_2 \sin px \quad \text{and} \quad T = C_3 e^{-p\alpha^2 t}$$

**Case 3**: $k=0$

Then equations (4) and (5) become...
The solutions of these equations are
\[ X = C_1 x + C_2 \quad \text{and} \quad T = C_3 \]
Since \( u(x, t) = X.T \) is the solution of Eq.(1), the three mathematically possible solutions of Eq.(1) are
\[ u(x, t) = (Ae^{px} + Be^{-px})e^{p^2 \alpha^2 t} \quad \text{..........................}(6) \]
\[ u(x, t) = (A \cos px + B \sin px)e^{-p^2 \alpha^2 t} \quad \text{..........................}(7) \]
and \[ u(x, t) = Ax + B \quad \text{..........................}(8) \]
where \( C_1, C_3 \) and \( C_2 C_3 \) have been taken as A and B.

**PROBLEMS**

1. Find the temperature distribution in a homogeneous bar of length \( \pi \) which is insulated laterally, if the ends are kept at zero temperature and if, initially, the temperature is \( k \) at the centre of the bar and falls uniformly to zero at its ends.

Solution:

Figure 4.3 represents the graph of the initial temperature in the bar.

Equation of OA is \( y = \frac{2k}{\pi} x \) and the equation of AB is \( \frac{y - 0}{k - 0} = \frac{x - \pi}{\frac{\pi}{2} - \pi} \)

i.e., \[ y = \frac{2k}{\pi} (\pi - x) \]

Hence \[ u(x, 0) = \begin{cases} 
2k & \text{in } 0 \leq x \leq \frac{\pi}{2} \\
0 & \text{in } \frac{\pi}{2} \leq x \leq \pi 
\end{cases} \]

The temperature distribution \( u(x, t) \) in the bar is given by
\[ \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial t^2} \quad \text{..........................}(1) \]
We have to solve Eq.(1) satisfying the following boundary conditions.

\[ u(0, t) = 0, \quad \text{for all } t \geq 0 \]  \hspace{1cm} ...................(2)

\[ u(\pi, t) = 0, \quad \text{for all } t \geq 0 \]  \hspace{1cm} ...................(3)

\[ u(x, 0) = \begin{cases} 
\frac{2k}{\pi} x, & \text{in } 0 \leq x \leq \frac{\pi}{2} \\
\frac{2k}{\pi} (\pi - x), & \text{in } \frac{\pi}{2} \leq x \leq \pi 
\end{cases} \]  \hspace{1cm} ...................(4)

As \( u(x, t) \) has to remain finite when \( t \to \infty \), the proper solution of Eq.(1) is

\[ u(x, t) = (A \cos px + B \sin px)e^{-pt^2/\alpha^2} \]  \hspace{1cm} ...................(5)

Using boundary condition (2) in (5), we have

\[ Ae^{-pt^2/\alpha^2} = 0, \quad \text{for all } t \geq 0 \]

\[ A = 0 \]

Using boundary condition (3) in (5), we have

\[ B \sin p\pi e^{-pt^2/\alpha^2} = 0, \quad \text{for all } t \geq 0 \]

\[ B = 0 \quad \text{or} \quad \sin p\pi = 0 \]

\[ p\pi = n\pi \quad \text{or} \quad p = n, \quad \text{where} \quad n = 0, 1, 2, \ldots, \infty \]

Using these values of \( A \) and \( p \) in (5), it reduces to

\[ u(x, t) = B \sin nx e^{-pt^2/\alpha^2} \]  \hspace{1cm} ...................(6)

where \( n = 1, 2, 3, \ldots, \infty \)

Therefore the most general solution of Eq.(1) is

\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin nx e^{-pt^2/\alpha^2} \]  \hspace{1cm} ...................(7)

Using boundary condition (4) in (7), we have

\[ \sum_{n=1}^{\infty} B_n \sin nx = f(x) \quad \text{in} \quad (0, \pi), \quad \text{where} \]

\[ f(x) = \begin{cases} 
\frac{2k}{\pi} x, & \text{in } 0 \leq x \leq \frac{\pi}{2} \\
\frac{2k}{\pi} (\pi - x), & \text{in } \frac{\pi}{2} \leq x \leq \pi 
\end{cases} \]

If the Fourier half-range sine series of \( f(x) \) in \( (0, \pi) \) is \( \sum_{n=1}^{\infty} B_n \sin nx \), it is comparable with \( \sum_{n=1}^{\infty} B_n \sin nx \).

Hence \( B_n = b_n = \frac{2}{\pi} \left[ \frac{\pi}{2} \int_{0}^{\frac{\pi}{2}} x \sin nx \, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{2k}{\pi} (\pi - x) \sin nx \, dx \right] \)
Using this value in (7), the required solution is
\[ u(x, t) = \frac{8k}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin nx e^{-n^2 \alpha^2 t} \]
\[ u(x, t) = \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin(2n-1) x e^{-(2n-1)^2 \alpha^2 t} \]

2. Solve the one dimensional heat flow equation
\[ \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \]
satisfying the following boundary conditions.

(i) \[ \frac{\partial u}{\partial x}(0, t) = 0, \quad \text{for all } t \geq 0 \]
(ii) \[ \frac{\partial u}{\partial x}(\pi, t) = 0, \quad \text{for all } t \geq 0; \text{ and} \]
(iii) \[ u(x, 0) = \cos^2 x, \quad 0 < x < \pi \]

Solution:

The appropriate solution of the equation
\[ \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \]  

\[ u(x, t) = (A \cos px + B \sin px) e^{-\rho^2 \alpha^2 t} \]  

Differentiating (2) partially w.r.t. x, we have
\[ \frac{\partial u}{\partial x}(x, t) = p(-A \sin px + B \cos px) e^{-\rho^2 \alpha^2 t} \]

Using boundary condition (i) in (3), we have
\[ pB e^{-\rho^2 \alpha^2 t} = 0, \quad \text{for all } t \geq 0 \]
\[ \therefore \quad B = 0 \]

\[ \text{[\text{if } p = 0, u(x, t) = A, \quad \text{which is meaningless}] \]

Using boundary condition (ii) in (3), we have
\[ -pA \sin p\pi e^{-\rho^2 \alpha^2 t} = 0, \quad \text{for all } t \geq 0 \]
\[ \therefore \quad \text{Either } A = 0 \quad \text{or} \quad \sin p\pi = 0 \]

A = 0 leads to a trivial solution.
\[ \sin p\pi = 0 \]
\[ p\pi = n\pi \quad \text{or} \quad p = n, \quad \text{where } n = 0, 1, 2, \ldots, \infty \]

Using these values of B and p in (2), it reduces to
\[ u(x, t) = A \cos nx e^{-\rho^2 \alpha^2 t} \]

\[ \therefore \quad \text{4} \]
where $n = 0, 1, 2, \ldots \infty$

Therefore the most general solution of Eq.(1) is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos nx e^{-\alpha^2 t}$$

Using boundary condition (iii) in (5), we have

$$\sum_{n=0}^{\infty} A_n \cos nx = \cos^2 x \text{ in } (0, \pi)$$

In general, we have to expand the function in the R.H.S. as a Fourier half-range cosine series in $(0, \pi)$ so that it may be compared with L.H.S. series.

In this problem, it is not necessary. We can rewrite $\cos^2 x$ as $\frac{1}{2}(1 + \cos 2x)$, so that comparison is possible.

Thus

$$\sum_{n=0}^{\infty} A_n \cos nx = \frac{1}{2} + \frac{1}{2} \cos 2x$$

Comparing like terms, we have

$$A_0 = \frac{1}{2}, \quad A_2 = \frac{1}{2}, \quad A_1 = A_3 = A_4 = \ldots = 0$$

Using these values in (5), the required solution is

$$u(x,t) = \frac{1}{2} + \frac{1}{2} \cos 2x e^{-\alpha^2 t}$$

3. Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ satisfying the following conditions.

(i) $u$ is finite when $t \to \infty$.

(ii) $\frac{\partial u}{\partial x} = 0$ when $x = 0$, for all values of $t$.

(iii) $u = 0$ when $x = l$, for all values of $t$.

(iv) $u = u_0$ when $t = 0$, for $0 < x < l$.

Solution:

We have to solve the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

satisfying the following boundary conditions.

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \text{for all } t \geq 0 \quad \text{..................(2)}$$

$$u(l, t) = 0, \quad \text{for all } t \geq 0 \quad \text{..................(3)}$$

$$u(x, 0) = u_0 \quad \text{for } 0 < x < l. \quad \text{..................(4)}$$

Since $u$ is finite when $t \to \infty$, the proper solution of Eq.(1) is

$$u(x,t) = (A \cos px + B \sin px)e^{-\rho^2 \alpha^2 t} \quad \text{..................(5)}$$

Differentiating (5) partially w.r.t. $x$, we have
\[ \frac{\partial u}{\partial x}(x, t) = p(-A \sin px + B \cos px)e^{-p^2 \alpha^2 t} \]  

\[ \text{Using boundary condition (ii) in (6), we have} \]
\[ p.B.e^{-p^2 \alpha^2 t} = 0, \; \text{for all } t \geq 0 \]
\[ \therefore \quad B = 0 \quad \text{[if } p = 0, u(x, t) = A, \; \text{which is meaningless]} \]

\[ \text{Using boundary condition (iii) in (5), we have} \]
\[ -A \cos pl.e^{-p^2 \alpha^2 t} = 0, \; \text{for all } t \geq 0 \]
\[ \therefore \quad \text{Either } A = 0 \; \text{or } \cos pl = 0 \]
\[ A = 0 \text{ leads to a trivial solution.} \]
\[ \cos pl = 0 \]
\[ pl = \text{an odd multiple of } \frac{\pi}{2} \; \text{or } (2n-1) \frac{\pi}{2} \]
\[ pl = \frac{(2n-1)\pi}{2l}, \; \text{where } n = 0, 1, 2, \ldots \infty. \]

\[ \text{Using these values of } B \text{ and } p \text{ in (5), it reduces to} \]
\[ u(x, t) = A \cos \frac{(2n-1)\pi x}{2l} e^{-p^2 \alpha^2 t/4l^2} \]
\[ \text{where } n = 1, 2, 3, \ldots \infty. \]
\[ \text{Therefore the most general solution of Eq.(1) is} \]
\[ u(x, t) = \sum_{n=1}^{\infty} A_{2n-1} \cos \frac{(2n-1)\pi x}{2l} e^{-p^2 \alpha^2 t/4l^2} \]
\[ \text{Using boundary condition (iv) in (8), we have} \]
\[ \sum_{n=1}^{\infty} A_{2n-1} \cos \frac{(2n-1)\pi x}{2l} = u_0 \text{ in } (0, l) \]
\[ \text{The series in the L.H.S of (9) is not in the form of the Fourier half-range cosine series of any function in } (0, l), \text{ that is, } \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}. \text{ Hence, to find } A_{2n-1}, \text{ we proceed as in the derivation of Euler’s formula for the Fourier coefficients.} \]

\[ \text{Multiplying both sides of (9) by } \cos \frac{(2n-1)\pi x}{2l} \text{ and integrating w.r.t. x between 0 and } l, \text{ we get} \]
\[ A_{2n-1} \int_0^l \cos^2 \frac{(2n-1)\pi x}{2l} dx = u_0 \int_0^l \cos \frac{(2n-1)\pi x}{2l} dx \]
\[ \text{[}\therefore \text{All other integrals in the L.H.S. vanish]} \]
\[ A_{2n-1} \int_0^l \left[ x \cos \frac{(2n-1)\pi x}{2l} \right]_0^l = u_0 \left[ \sin \frac{(2n-1)\pi x}{2l} \right]_0^l \]
Using this value in (8), the required solution is
\[ u(x, t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos \left( \frac{(2n-1)\pi x}{2l} \right) e^{-\alpha_k^2 t^2/4t^2} \]

PROBLEMS ON TEMPERATURE IN A SLAB WITH FACES WITH ZERO TEMPERATURE
1. Faces of a slab of width c are kept at temperature zero. If the initial temperature in the slab is \( f(x) \), determine the temperature formula. If \( f(x) = u_0 \), a constant, find the flux \(-k \frac{\partial u}{\partial x}(x_0, t)\) across any plane \( x = x_0 \) \((0 \leq x_0 \leq c)\) and show that no heat flows across the central plane \( x_0 = \frac{c}{2} \), where \( k^2 \) is the diffusivity of the substance.

Solution:

\[ \frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \] ..........................(1)

We have to solve Eq.(1) satisfying the following boundary conditions.
\[ u(0, t) = 0, \text{ for all } t \geq 0 \] ..........................(2)
\[ u(c, t) = 0, \text{ for all } t \geq 0 \] ..........................(3)
\[ u(x, 0) = f(x) \text{ for } 0 < x < c \] ..........................(4)

Proceeding as before, the most general solution of Eq.(1) is
\[ u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} e^{-\eta^2 n^2 k^2 t / c^2} \] ............................(5)

Using boundary condition (4) in (5), we have
\[ \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} = f(x) \text{ in } (0,c) = \sum b_n \sin \frac{n\pi x}{c} \]
which is the Fourier half-range sine series of \( f(x) \) in \((0,c)\).
Comparing like terms, we get
\[ B_n = b_n = \frac{2}{c} \int_{0}^{c} f(x) \sin \frac{n\pi x}{c} \, dx \] ............................(6)

Using (6) in (5), the required solution is
\[ u(x,t) = \frac{2}{c} \sum_{n=1}^{\infty} \int_{0}^{c} f(\theta) \sin \frac{n\pi \theta}{c} \, d\theta \] ............................(7)

When \( f(x) = u_0 \), from (6), we get
\[ B_n = \frac{2}{c} \int_{0}^{c} u_0 \sin \frac{n\pi x}{c} \, dx \]
\[ = \frac{2u_0}{c} \left( \cos \frac{n\pi x}{c} \right)_{0}^{c} \]
\[ = \frac{2u_0}{c} (1 - \cos n\pi) \]
\[ = \begin{cases} 4u_0/n\pi, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \]

Therefore the required solution in this case is
\[ u(x,t) = \frac{4u_0}{c} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{c} e^{-(2n-1)^2 \eta^2 k^2 t / c^2} \] ............................(8)

Differentiating (8) partially w.r.t \(x\),
\[ \frac{\partial u}{\partial x}(x,t) = \frac{4u_0}{c} \sum_{n=1}^{\infty} \cos \frac{(2n-1)\pi x}{c} e^{-(2n-1)^2 \eta^2 k^2 t / c^2} \]

Therefore the flux across the plane \( x = x_0 \) is given by
\[ -k \frac{\partial u}{\partial x}(x_0,t) = -\frac{4ku_0}{c} \sum_{n=1}^{\infty} \cos \frac{(2n-1)\pi x_0}{c} e^{-(2n-1)^2 \eta^2 k^2 t / c^2} \]
Therefore the flux across the central plane \( x = \frac{c}{2} \) is given by

\[-k \frac{\partial u}{\partial x} \left( \frac{c}{2}, t \right) = - \frac{4k u_0}{c} \sum_{n=1}^{\infty} \cos \frac{(2n-1)\pi}{2} e^{-\left(\frac{2n-1}{2}\pi \frac{c^2}{2} / c^2 \right)}
\]

That is no heat flow across the central plane of the slab.

PROBLEMS WITH NON-ZERO BOUNDARY VALUES (TEMPERATURE OR TEMPERATURE GRADIENTS)

1. A bar AB with insulated sides is initially at temperature 0°C throughout.

Heat is suddenly applied at the end \( x = l \) at a constant rate \( A \), so that \( \frac{\partial u}{\partial x} = A \) for \( x = l \), while the end A is not disturbed. Find the subsequent temperature distribution in the bar.

Solution:

The temperature distribution \( u(x, t) \) in the bar is given by the equation.

\[ \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \]  

We have to solve Eq.(1) satisfying the following boundary conditions.

\[ u(0, t) = 0, \quad \text{for all } t \geq 0 \]  

\[ \frac{\partial u}{\partial x}(l, t) = A, \quad \text{for all } t \geq 0 \]  

\[ u(x, 0) = 0 \quad \text{for } 0 < x < l. \]  

Since condition (3) has a non-zero value on the right side, we adopt the modified procedure.

Let \( u(x, t) = u_1(x) + u_2(x, t) \)  

Where \( u_1(x) \) is given by

\[ \frac{d^2 u_1}{dx^2} = 0 \]  

and \( u_2(x, t) \) is given by

\[ \frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \]  

The boundary conditions for Eq.(6) are

\[ u_1(0) = 0 \]  

\[ u_1(l) = A, \]  

\[ u_2(x, 0) = 0 \quad \text{for } 0 < x < l. \]  

\[ \frac{\partial u_2}{\partial x}(l, t) = A, \quad \text{for all } t \geq 0 \]  

\[ u_2(x, t) = 0 \quad \text{for } x = 0 \]  

\[ u_2(x, t) = 0 \quad \text{for } x = l. \]  

\[ \frac{\partial u_2}{\partial t}(0, t) = 0 \quad \text{for } 0 < x < l, \]  

\[ \frac{\partial u_2}{\partial t}(l, t) = 0 \quad \text{for } 0 < x < l. \]  

\[ \frac{\partial u_2}{\partial t}(x, 0) = 0 \quad \text{for } 0 < x < l. \]
and \[ \frac{du_1}{dx}(l) = A \] ..........................(9)

Solving Eq.(6), we get \[ u_1(x) = C_1 x + C_2 \] ..........................(10)

Using boundary condition (8) in (10), we get \[ C_2 = 0 \]

From (10), we have \[ \frac{du_1}{dx}(x) = C_1 \] ..........................(11)

Using boundary condition (9) in (11), we get \[ C_1 = A \]

\[ u_1(x) = Ax \] ..........................(12)

The boundary conditions for Eq.(7) are

\[ u_2(0,t) = u(0,t) - u_1(0) = 0, \quad \text{for all} \ t \geq 0 \] ..........................(13)

\[ \frac{\partial u_2}{\partial x}(l,t) = \frac{\partial u}{\partial x}(l,t) - \frac{\partial u_1}{\partial x}(l) = 0, \quad \text{for all} \ t \geq 0 \] ..........................(14)

\[ u_2(x,0) = u(x,0) - u_1(x) = -Ax, \quad \text{for} \ 0 < x < l \] ..........................(15)

Proceeding as before, we get the most general solution of Eq.(7) as

\[ u_2(x,t) = \sum_{n=1}^{\infty} B_{2n-1} \sin \left( \frac{(2n-1)\pi x}{2l} \right) \exp \left\{ -\frac{(2n-1)^2 \pi^2 \alpha^2 t}{4l^2} \right\} \] ..........................(16)

Using boundary condition (15) in (16), we have

\[ \sum_{n=1}^{\infty} B_{2n-1} \sin \left( \frac{(2n-1)\pi x}{2l} \right) = -Ax \ 	ext{in} \ (0,l) \]

\[ \therefore \ B_{2n-1} = \frac{2}{l} \int_{0}^{l} Ax \sin \left( \frac{(2n-1)\pi x}{2l} \right) \, dx \]

\[ = \frac{2A}{l} \left[ x \left\{ -\cos \left( \frac{(2n-1)\pi x}{2l} \right) \right\} \right]_{0}^{l} - \left\{ \frac{\sin \left( \frac{(2n-1)\pi x}{2l} \right)}{\frac{2l}{(2n-1)^2 \pi^2}} \right\} \left\{ \frac{\sin \left( \frac{(2n-1)\pi x}{2l} \right)}{\frac{2l}{(2n-1)^2 \pi^2}} \right\} \right\} \]

\[ = - \frac{8Al}{(2n-1)^2 \pi^2} \sin \left( \frac{(2n-1)\pi x}{2l} \right) \]

\[ = \frac{8Al(-1)^n}{(2n-1)^2 \pi^2} \]

Using this value in (16) and then using (12) and (16) in (5), the required solution is

\[ u(x,t) = \sum_{n=1}^{\infty} B_{2n-1} \sin \left( \frac{(2n-1)\pi x}{2l} \right) \exp \left\{ -\frac{(2n-1)^2 \pi^2 \alpha^2 t}{4l^2} \right\} \]
**Steady State Heat Flow in Two Dimensions:**

**Variable Separable Solutions of Laplace Equation:**

Laplace equation in two dimensional Cartesians is

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]  \hspace{1cm} \ldots \ldots (1)

Let

\[ u(x, y) = X(x)Y(y) \]  \hspace{1cm} \ldots \ldots (2)

be the solution of the equation (1).

Where \( X(x) \) is a function of ‘\( x \)’ alone and \( Y(y) \) is a function of ‘\( y \)’ alone.

Then

\[ \frac{\partial^2 u}{\partial x^2} = X''Y \text{ and } \frac{\partial^2 u}{\partial y^2} = XY'' \], Where \( X'' = \frac{d^2 X}{dx^2} \text{ and } Y'' = \frac{d^2 Y}{dy^2} \) Satisfy equation (1)

i.e

\[ X''Y + XY'' = 0 \]

The L.H.S of (3) is a function of \( x \) alone and the R.H.S is a function of \( t \) alone. They are equal for all values of the independent variable \( x \) and \( t \). This is possible only if each is a constant.

\[ \frac{X''}{X} = -\frac{Y''}{Y} = k \]  \hspace{1cm} \ldots \ldots (3)

Where \( k \) is a constant.

\[ X'' - kX = 0 \]  \hspace{1cm} \ldots \ldots (4)

and

\[ Y'' + kY = 0 \]  \hspace{1cm} \ldots \ldots (5)

The nature of the solution of (4) and (5) depends on the nature of values of \( k \). Hence the following three cases arise.

**Case 1:**

\( k \) is positive. Let \( k = p^2 \)

Then equation (4) and (5) become

\[ (D^2 - p^2)X = 0 \]

and

\[ (D'^2 + p^2)Y = 0 \]

Where

\[ D = \frac{d}{dx} \text{ and } D' = \frac{d}{dy} \]

The solutions of these equations are

\[ X = Ae^{px} + Be^{-px} \]

and
$Y = C \cos py + D \sin py$

**Case 2:**
k is negative. Let $k = -p^2$

Then equation (4) and (5) become

$\left( D^2 + p^2 \right) X = 0$

and

$\left( D^2 - p^2 \right) Y = 0$

The solutions of these equations are

$X = A \cos px + B \sin px$

and

$Y = Ce^{py} + De^{-py}$

**Case 3:**
k = 0.

Then equation (4) and (5) become

$\frac{d^2 X}{dx^2} = 0$

and

$\frac{d^2 Y}{dy^2} = 0$

The solutions of these equations are

$X = Ax + B$

and

$Y = Cy + D$

Since $u(x, y) = X(x)Y(y)$ is the solution of the equation (1), the three mathematically possible solutions of the equation (1) are

$u(x, y) = \left( Ae^{px} + Be^{-px} \right) \left( C \cos py + D \sin py \right) \quad ........ (6)$

$u(x, y) = \left( A \cos px + B \sin px \right) \left( Ce^{py} + De^{-py} \right) \quad ........ (7)$

and

$u(x, y) = \left( Ax + B \right) \left( Cy + D \right) \quad ........ (8)$

**Problems:**

(1) A rectangular plane with insulated surface is a cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the two long edges x=0 and x=a and the short edge at infinity are kept at temperature $0^\circ C$, while the other short edge y=0 is kept at temperature (i) $u_0 \sin^3 \frac{\pi x}{a}$ and (ii) T (constant). **Find the steady state temperature at any point (x, y) of the plate.**
Solution:

The temperature $u(x, y)$ at any point $(x, y)$ of the plate in the steady state is given by the equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{............... (1)}$$

We have to solve equation (1) satisfying the following boundary conditions.

$$u(0, y) = 0, \quad \text{for} \quad y > 0 \quad \text{............... (2)}$$
$$u(a, y) = 0, \quad \text{for} \quad y > 0 \quad \text{............... (3)}$$
$$u(x, \infty) = 0, \quad \text{for} \quad 0 \leq x \leq a \quad \text{............... (4)}$$
$$u(x, 0) = f(x), \quad \text{for} \quad 0 \leq x \leq 2l \quad \text{............... (5)}$$

Where $f(x) = u_0 \sin^3 \frac{\pi x}{a}$ for (i) and $f(x) = T$ for (ii).

Three possible solutions of the equation (1) are

$$u(x, y) = \left(Ae^{px} + Be^{-px}\right) \left(C \cos py + D \sin py\right) \quad \text{............... (6)}$$
$$u(x, y) = \left(A \cos px + B \sin px\right) \left(Ce^{py} + De^{-py}\right) \quad \text{............... (7)}$$
and

$$u(x, y) = \left(Ax + B\right) \left(Cy + D\right) \quad \text{............... (8)}$$

By boundary condition (4), $u \to 0$ when $y \to \infty$. of the three possible solutions, only solution (7) can satisfy this condition. Hence we reject the other two solutions.

Rewriting (7), we have

$$u(x, y)e^{-py} = \left(A \cos px + B \sin px\right) \left(C + De^{-2py}\right) \quad \text{............... (9)}$$
Using boundary condition (4) in (9), we have
\[
(A \cos px + B \sin px)C = 0, \quad \text{for } 0 \leq x \leq a
\]
\[
C = 0
\]
Using boundary conditions (2) in (7), we have
\[
A.D e^{-ny} = 0 \quad \text{for all } y > 0
\]
Either \( A = 0 \) or \( D = 0 \)
If we assume that \( D = 0 \), we get a trivial solution \( A = 0 \)

Using boundary conditions (3) in (7), we have
\[
B \sin pa.D e^{-ny} = 0 \quad \text{for all } y > 0
\]
The assumption that \( B = 0 \) leads to a trivial solution.
\[
\sin pa = 0
\]
\[
pa = n\pi
\]
\[
p = \frac{n\pi}{a}
\]
Where \( n = 0, 1, 2, \ldots \infty \)
Using these values of \( A, p, C \) in (7), the solution reduces to
\[
u(x, y) = \lambda \sin \frac{n\pi x}{a} e^{-\frac{ny}{a}}
\]  \quad \text{.........(10)}

where \( n = 0, 1, 2, 3, \ldots \infty \)
The most general solution of Eq. (1) is
\[
u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{a} e^{-\frac{ny}{a}}
\]  \quad \text{.........(11)}
Using boundary conditions (5) in (11), we have
\[
\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{a} = f(x) \quad \text{in } (0, a)
\]  \quad \text{.........(12)}

(i) \( f(x) = u_0 \sin \frac{n\pi x}{a} \)
\[
u(x, y) = \frac{u_0}{4} \left( 3 \sin \frac{\pi x}{a} - \sin \frac{3\pi x}{a} \right)
\]
Using this form of \( f(x) \) in (12) and comparing like terms, we get
\[
\lambda_1 = \frac{3u_0}{4}, \lambda_3 = -\frac{u_0}{4}, \text{ and } \lambda_2 = 0 = \lambda_4 = \ldots
\]
Using these values in (11), the required solution is
\[
u(x, y) = \frac{3u_0}{4} \sin \frac{\pi x}{a} e^{-\frac{ny}{a}} - \frac{u_0}{4} \sin \frac{3\pi x}{a} e^{-\frac{3ny}{a}}
\]

(ii) \( f(x) = T \quad \text{in } (0, a) \)
Let the Fourier half-range sine series of \( f(x) \) in \((0, a)\) be \( \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \)

Using this form of \( f(x) \) in (12) and comparing like terms, we get

\[
\lambda_n = b_n = \frac{2}{a} \int_0^a T \sin \frac{n\pi x}{a} \, dx
\]

\[
= \frac{2T}{a} \left\{ \frac{-\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right\}^a_0
\]

\[
= \frac{2T}{n\pi} \left\{ -(-1)^n \right\}
\]

\[
= \begin{cases} 
0, & \text{if } n \text{ is even} \\
\frac{4T}{n\pi}, & \text{if } n \text{ is odd} 
\end{cases}
\]

Using this value of \( \lambda_n \) in (11), the required solution is

\[
u(x, y) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin \frac{(2n-1)\pi x}{a} \exp \left( \frac{(2n-1)\pi y}{a} \right).
\]

(2) An infinitely long metal plate in the form of an area is enclosed between the lines \( y = 0 \) and \( y = \pi \) for positive values of \( x \). The temperature is zero along the edges \( y = 0, y = \pi \) and the edge at infinity. If the edge \( x = 0 \) is kept at temperature \( ky \), find the steady state temperature distribution in the plate.

Solution:

![Fig.2](image)

The temperature \( u(x, y) \) at any point \((x, y)\) of the plate in the steady state is given by the equation.
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{……………… (1)}
\]

We have to solve equation (1) satisfying the following boundary conditions.

\[
u(x,0) = 0, \quad \text{for} \quad x > 0 \quad \text{……………… (2)}
\]

\[
u(x,\pi) = 0, \quad \text{for} \quad x > 0 \quad \text{……………… (3)}
\]

\[
u(\infty,y) = 0, \quad \text{for} \quad 0 \leq y \leq \pi \quad \text{……………… (4)}
\]

\[
u(0,y) = ky, \quad \text{for} \quad 0 \leq y \leq \pi \quad \text{……………… (5)}
\]

Of the three possible solutions of Eq.(1), the solution

\[
u(x,y) = \left( A e^{px} + Be^{-px} \right) (C \cos py + D \sin py) \quad \text{……………… (6)}
\]

can satisfy the boundary condition (4). Rewriting (6), we have

\[
u(x,y) e^{-px} = \left( A + Be^{-2px} \right) (C \cos py + D \sin py) \quad \text{……………… (7)}
\]

Using boundary condition (4) in (6), we have

\[
A(C \cos py + D \sin py) = 0, \quad \text{for} \quad 0 \leq y \leq \pi
\]

\[
A = 0
\]

Using boundary conditions (2) in (6), we have

\[
B,C e^{px} = 0 \quad \text{for all} \quad x > 0
\]

Either \( B = 0 \) or \( C = 0 \)

If we assume that \( B=0 \), we get a trivial solution \( C=0 \)

Using boundary conditions (3) in (6), we have

\[
D \sin p\pi, Be^{py} = 0 \quad \text{for all} \quad x > 0
\]

\[
B = 0, D = 0 \text{ or } \sin p\pi = 0
\]

The values \( B=0 \) and \( D=0 \) leads to a trivial solution.

\[
\sin p\pi = 0
\]

\[
p = n
\]

Where \( n = 0,1,2,\cdots \infty \)

Using these values of \( A, p, C \) in (6), the solution reduces to

\[
u(x,y) = \lambda e^{\pi x} \sin ny, \quad \text{where} \quad n = 0,1,2,\cdots \infty \quad \text{……………… (8)}
\]

The most general solution of Eq.(1) is

\[
u(x,y) = \sum_{n=1}^{\infty} \lambda_n e^{-\pi x} \sin ny \quad \text{……………… (9)}
\]

Using boundary conditions (5) in (7), we have
\[ \sum_{n=1}^{\infty} \lambda_n \sin ny = ky \quad \text{in } (0, \pi) \]

\[ = \sum b_n \sin ny \]

Which is the Fourier half-range sine series of \( ky \) in \( (0, \pi) \). Comparing like terms in the two series, we get.

\[ \lambda_n = b_n = \frac{2}{\pi} \int_0^\pi ky \sin ny \, dy \]

\[ = \frac{2k}{\pi} \left( y \left( \frac{-\cos ny}{n^2} \right) \left( \frac{-\sin ny}{n^2} \right) \right)_0^\pi \]

\[ = \frac{2k}{n} (-1)^{n+1} \]

Using this value of \( \lambda_n \) in (7), the required solution is

\[ u(x, y) = 2k \sum \frac{(-1)^{n+1}}{n} e^{-nx} \sin ny \]

(3) A rectangular plate with insulated surface is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge \( x = 0 \) is given by

\[ u = \begin{cases} 10y, & \text{for } 0 \leq y \leq 10 \\ 10(20 - y), & \text{for } 10 \leq y \leq 20 \end{cases} \]

and the two long edges as well as the other short edge are kept at \( 0^\circ \text{C} \). Find the steady state temperature distribution in the plate.

Solution:

The steady state temperature \( u(x, y) \) at any point \((x, y)\) of the plate in the steady state is given by the equation..

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{……. (1)} \]

We have to solve equation (1) satisfying the following boundary conditions.

\[ u(x, 0) = 0, \quad \text{for } x > 0 \quad \text{……. (2)} \]
\[ u(x, 20) = 0, \quad \text{for } x > 0 \quad \text{……. (3)} \]
\[ u(\infty, y) = 0, \quad \text{for } 0 \leq y \leq 20 \quad \text{……. (4)} \]
\[ u(0, y) = f(y), \quad \text{for } 0 \leq y \leq 20 \quad \text{……. (5)} \]

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where

\[ f(y) = \begin{cases} 10y, & \text{for } 0 \leq y \leq 10 \\ 10(20 - y), & \text{for } 10 \leq y \leq 20 \end{cases} \]

The most general solution of Eq. (1) is

\[ u(x, y) = \sum_{n=1}^{\infty} \lambda_n e^{-ny/20} \sin \frac{n\pi y}{20} \] ............ (6)

Using boundary conditions (5) in (6), we have

\[ \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi y}{20} = f(y) \text{ in (0,20)} \]

\[ = \sum b_n \sin \frac{n\pi y}{20} \]

Which is Fourier half-range sine series of \( f(y) \) in (0,20)

Comparing like terms, we get

\[ \lambda_n = b_n = \frac{2}{20} \int_{0}^{20} f(x) \sin \frac{n\pi x}{20} \, dx, \text{ by Euler's formula} \]

\[ = \int_{0}^{10} y \sin \frac{n\pi y}{20} \, dy + \int_{10}^{20} (20-y) \sin \frac{n\pi y}{20} \, dy \]

\[ = \left\{ y \left( -\frac{\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - \left( -\sin \frac{n\pi y}{20} \right) \right\}_{0}^{10} + \left\{ (20-y) \left( -\frac{\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - \left( -\sin \frac{n\pi y}{20} \right) \right\}_{10}^{20} \]

\[ = \left\{ \left( -\frac{200}{n\pi} \cos \frac{n\pi}{2} + \frac{400}{n^2\pi^2} \sin \frac{n\pi}{2} \right) + \left( \frac{200}{n\pi} \cos \frac{n\pi}{2} + \frac{400}{n^2\pi^2} \sin \frac{n\pi}{2} \right) \right\} \]

\[ = \frac{800}{n^2\pi^2} \sin \frac{n\pi}{2} \]

Using this value of \( \lambda_n \) in (6), the required solution is

\[ u(x, y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \exp \left\{ \frac{(2n-1)\pi x}{20} \right\} \sin \frac{(2n-1)\pi y}{20} \]
(4) A long rectangular plate with insulated surface is 1 cm wide. If the temperature along one short edge (y=0) is \( u(x,0) = k(2lx - x^2) \) degrees, for 0<x<l, while the two long edges x=0 and x=l as well as the other short edge are kept at temperature 0\(^\circ\) C, Find the steady state temperature function \( u(x, y) \).

**Solution:**

The steady state temperature \( u(x, y) \) at any point \((x, y)\) of the plate in the steady state is given by the equation.

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{............... (1)}
\]

We have to solve equation (1) satisfying the following boundary conditions.

\[
\begin{align*}
    u(0, y) & = 0, \quad \text{for} \quad y > 0 \quad \text{............... (2)} \\
    u(a, y) & = 0, \quad \text{for} \quad y > 0 \quad \text{............... (3)} \\
    u(x, \infty) & = 0, \quad \text{for} \quad 0 \leq x \leq l \quad \text{............... (4)} \\
    u(x, 0) & = k(2lx - x^2), \quad \text{for} \quad 0 \leq x \leq l \quad \text{............... (5)}
\end{align*}
\]

The most general solution of Eq.(1) is

\[
u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \quad \text{............... (6)}
\]

Using boundary condition (5) in (6), we have

\[
\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = k(2lx - x^2) \quad \text{in} \quad (0, l)
\]

\[
= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}
\]

Which is Fourier half-range sine series of \( k(2lx - x^2) \) in \((0, l)\).

Comparing like terms, we get

\[
\lambda_n = b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx, \text{ by Euler's formula}
\]

\[
= \frac{2}{l} \int_{0}^{l} k(2lx - x^2) \sin \frac{n\pi x}{l} \, dx
\]
Using this value of $\lambda_n$ in (6), the required solution is

$$u(x,t) = \frac{8kl^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin \left( \frac{(2n-1)\pi x}{l} \right) \exp \left\{ \left( \frac{(2n-1)\pi y}{l} \right)^2 \right\}$$
UNIT 3
PART A

1. Classify the partial differential equation \(3u_{xx} + 4u_{xy} + 3u_{y} - 2u_{x} = 0\).
   Ans:
   Given \(3u_{xx} + 4u_{xy} + 3u_{y} - 2u_{x} = 0\).
   \(A = 3, \ B = 4, \ C = 0\)
   \(B^2 - 4AC = 16 > 0\),   Hyperbolic.

2. The ends A and B of a rod of length 10 cm long have their temperature kept at \(20^\circ C\) and \(70^\circ C\). Find the steady state temperature distribution on the rod.
   Ans:
   When the steady state conditions exists the heat flow equation is
   \[\frac{\partial^2 u}{\partial x^2} = 0\]
   i.e.,   \(u(x) = c_1x + c_2\)  
   The boundary conditions are  (a) \(u(0) = 20\),   (b) \(u(10) = 70\)
   Applying (a) in (1), we get
   \(u(0) = c_2 = 20\)
   Substitute this value in (1), we get
   \(u(x) = c_1x + 20\)
   ...................(2)
   Applying (b) in (2), we get
   \(u(10) = c_110 + 20 = 70\)
   \(\therefore c_1 = 5\)
   Substitute this value in (2), we get
   \(u(x) = 5x + 20\)

3. Solve the equation \(3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0\), given that \(u(x,0) = 4e^{-x}\) by the method of separation of variables.
   Ans:
   Given \(3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0\)
   .....................(1)
   Let \(u = X(x).Y(y)\)
   ......................(2)
   Be a solution of (1)
   \[\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY'\]
   ......................(3)
Substituting (3) in (1) we get
\[3X’Y + 2XY’ = 0\]
\[\frac{3X’}{X} = \frac{2Y’}{Y} = K\]
\[3X’ - KX = 0, \quad -2Y’ - KY = 0\]
\[3\frac{dX}{dx} = KX, \quad 2\frac{dY}{dy} = -KY\]
\[3\frac{dX}{dx} = K \cdot dx, \quad 2\frac{dY}{dy} = -K \cdot dy\]
Integrating we get
\[3\log X = Kx, \quad 2\log Y = Ky\]
\[X = e^{\frac{kx}{3}}, \quad Y = e^{\frac{ky}{2}}\]
Therefore \( u = X \cdot Y = e^{\frac{kx}{3}} \cdot e^{\frac{ky}{2}} \)

4. Write the one dimensional wave equation with initial and boundary conditions in which the initial position of the string is \( f(x) \) and the initial velocity imparted at each point \( x \) is \( g(x) \).

\textbf{Ans:}

The one dimensional wave equation is \( \frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2} \)

The boundary conditions are
\[(i) \ y(0, t) = 0 \quad (ii) \ y(x, 0) = f(x) \quad (iii) \ y(l, t) = 0 \quad (iv) \ \frac{\partial y(x,0)}{\partial t} = g(x)\]

5. What is the basic difference between the solution of one dimensional wave equation and one dimensional heat equation.

\textbf{Ans:}

Solution of the one dimensional wave equation is of periodic in nature. But solution of the one dimensional heat equation is not of periodic in nature.

6. In steady state conditions derive the solution of one dimensional heat flow equation.

\textbf{Ans:}

When steady state conditions exist the heat flow equation is independent of time \( t \).
\[\therefore \ \frac{\partial u}{\partial t} = 0\]
The heat flow equation becomes
\[\frac{\partial^2 u}{\partial x^2} = 0\]

7. What are the possible solutions of one dimensional wave equation.

\textbf{Ans:}
8. In the wave equation \( \frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2} \) what does \( \alpha^2 \) stand for?

Ans: 
\[ \alpha^2 = \frac{T}{mass} \]


Ans: 
The rate at which heat flows across an area \( A \) at a distance \( x \) from one end of a bar given by 
\[ Q = -KA \left( \frac{\partial u}{\partial x} \right)_x \]

K is thermal conductivity and \( \left( \frac{\partial u}{\partial x} \right)_x \) means the temperature gradient at \( x \).

10. What is the constant \( \alpha^2 \) in the wave equation \( u_{tt} = \alpha^2 u_{xx} \).

Ans: 
\[ \alpha^2 = \frac{Tension}{mass} \]

11. State any two laws which are assumed to derive one dimensional heat equation.

Ans: 
(i) The sides of the bar are insulated so that the loss or gain of heat from the sides by conduction or radiation is negligible.
(ii) The same amount of heat is applied at all points of the face.

12. Classify the PDE \( u_{xx} + xu_{yy} = 0 \).

Ans: 
Here \( A = 1, \ B = x, \ C = 0 \)
\[ \therefore \quad B^2 - 4AC = x^2 \]
(i) Elliptic if \( x > 0 \)
(ii) Parabolic if \( x = 0 \)
(iii) Hyperbolic if \( x < 0 \)

13. Classify the PDE
(a) \( y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} + 2u_x - 3u = 0 \).
(b) \( y^2 u_{xx} + u_{yy} + u_x^2 + u_y^2 + 7 = 0 \).
Ans:
(a) Here $A = y^2$, $B = -2xy$, $C = x^2$

$\therefore B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$

$\therefore$ Parabolic

(b) Here $A = y^2$, $B = 0$, $C = 1$.

$\therefore B^2 - 4AC = -4y^2 < 0$.

$\therefore$ Elliptic.

14. An insulated rod of length 60 cm has its ends at A and B maintained at 30° C and 40° C respectively. Find the steady state solution.

Ans:
The heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

...................(1)

When the steady state condition exist the heat flow equation becomes

$$\frac{\partial^2 u}{\partial x^2} = 0$$

i.e., $u(x) = c_1x + c_2$ ...................................(2)

The boundary conditions are (a) $u(0) = 30$, (b) $u(l) = 40$

Applying (a) in (2), we get

$u(0) = c_2 = 30$

Substitute this value in (2), we get

$u(x) = c_1x + 30$ ...................................(3)

Applying (b) in (3), we get

$u(l) = c_1l + 30 = 40$

$\therefore c_1 = \frac{40 - 30}{l}$

Substitute this value in (3), we get

$u(x) = \frac{40x - 30}{l} x + 30$

15. Solve using separation of variables method $yu_x + xu_y = 0$.

Ans:
Given $yu_x + xu_y = 0$.

...................(1)

Let $u = X(x)Y(y)$ ...................................(2)

Be a solution of (1)
\[ \frac{\partial u}{\partial x} = X'Y', \quad \frac{\partial u}{\partial y} = XY'. \]

\[ \text{.................(3)} \]

Substituting (3) in (1) we get
\[ yX'Y' + x.XY' = 0 \]
\[ \frac{X'}{xX} = \frac{-Y'}{yY} = K \]
\[ X' = KxX, \quad -Y' = KyY \]
\[ \frac{dX}{dx} = KxX, \quad \frac{dY}{dy} = KyY \]
\[ \frac{dX}{X} = Kx \, dx, \quad \frac{dY}{Y} = Ky \, dy \]

Integrating we get
\[ \log X = k_1 \frac{x^2}{2} + k_1, \quad \log Y = -k_2 \frac{y^2}{2} + k_2 \]
\[ X = c_1 e^{k_1 x^2}, \quad Y = c_2 e^{k_2 y^2} \]

Therefore \[ u = X \cdot Y = c_1 c_2 e^{k_1 x^2 - k_2 y^2} \]

PART B

(1) A tightly stretched string with fixed end points \( x=0 \) and \( x = l \) is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity \( \lambda x (l - x) \), then show that
\[ y(x,t) = \frac{8\lambda l^3}{\pi^4 a \, n^4} \sum_{n=1,3,5} \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi at}{l} \]

(2) A rectangular plate is bounded by lines \( x=0, \ y=0, \ x=a, \ y=b \). It’s surface are insulated. The temperature along \( x=0 \) and \( y=0 \) are kept at \( 0^\circ C \) and others at \( 100^\circ C \). Find the steady state temperature at any point of the plate.

(3) A metal bar 10 cm. long, with insulated sides has its ends A and B kept at \( 20^\circ C \) and \( 40^\circ C \) respectively until steady state conditions prevail. The temperature at A is then suddenly raised to \( 50^\circ C \) and at the same instant that at B is lowered to \( 10^\circ C \). Find the subsequent temperature at any point of the bar at any time.

(4) A tightly stretched string of length \( l \) has its ends fastened at \( x=0, \ x = l \). The midpoint of the string is mean taken to height ‘b’ and then released from rest in that position. Find the lateral displacement of a point of the string at time ‘t’ from the instant of release.
(5) If a square plate is bounded by the lines $x = \pm a$ and $y = \pm a$ and three of its edges are kept at temperature $0^\circ C$, while the temperature along the edge $y=a$ is kept at $u = x + a$, $-a \leq x \leq a$, find the steady state temperature in the plate.

(6) A uniform string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form of the curve

(i) $y = k \sin^3 \left( \frac{\pi x}{l} \right)$ and

(ii) $y = kx(l-x)$

and then releasing it from this position at time $t=0$. Find the displacement of the point of the string at a distance $x$ from one end at time $t$.

(7) A long rectangular plate with insulated surface is 1 cm wide. If the temperature along one short edge ($y=0$) is $u(x,0) = k(2lx - x^2)$ degrees, for $0 < x < l$, while the two long edges $x=0$ and $x=l$ as well as the other short edge are kept at temperature $0^\circ C$, find the steady state temperature function $u(x,y)$.

(8) Find the temperature distribution in a homogeneous bar of length $\pi$ which is insulated laterally, if the ends are kept at zero temperature and if, initially, the temperature is $k$ at the centre of the bar and falls uniformly to zero at its ends.

(9) Solve the one dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ in $-l \leq x \leq l$, $t \geq 0$, given that $y(-l,t) = 0$, $y(l,t) = 0$, $\frac{\partial y}{\partial x}(x,0) = 0$ and $y(x,0) = \frac{b}{l}(l-|x|)$

(10) A rectangular plate with insulated surface is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x=0$ is given by

$$u = \begin{cases} 10y, & \text{for } 0 \leq y \leq 10 \\ 10(20-y), & \text{for } 10 \leq y \leq 20 \end{cases}$$

and the two long edges as well as the other short edge are kept at $0^\circ C$. Find the steady state temperature distribution in the plate.

(11) Solve the one dimensional heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Satisfying the following boundary conditions.
(12) A rectangular plane with insulated surface is a cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the two long edges x=0 and x=a and the short edge at infinity are kept at temperature 0°C, while the other short edge y=0 is kept at temperature (i) $u_0 \sin^3 \frac{\pi x}{a}$ and (ii) T (constant). Find the steady state temperature at any point (x, y) of the plate.

(13) A tightly stretched strings with fixed end points x=0 and x=50 is initially at rest in its equilibrium position. If it is said to vibrate by giving each point a velocity (i) $v = v_0 \sin \frac{\pi x}{50}$ and (ii) $v = v_0 \sin \frac{\pi x}{50} \cos \frac{2\pi x}{50}$, Find the displacement of any point of the string at any subsequent time.

(14) An infinitely long metal plate in the form of an area is enclosed between the lines y = 0 and y = \pi for positive values of x. The temperature is zero along the edges y = 0, y = \pi and the edge at infinity. If the edge x=0 is kept at temperature ky, Find the steady state temperature distribution in the plate.

(15) A taut string of length 2l, fastened at both ends, is disturbed from its position of equilibrium by imparting to each of its points an initial velocity of magnitude $k(2lx - x^2)$Find the displacement function $y(x,t)$.